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Table of Contents

Articles

	<u>MATHEMATICAL CONTENT ON STEM ACTIVITIES</u> <i>Aitzol Lasa, Jaione Abaurrea, Haritz Iribas</i>	333-346
	<u>CHARACTERISTICS OF STUDENTS' ABDUCTIVE REASONING IN SOLVING ALGEBRA PROBLEMS</u> <i>Indriati Nurul Hidayah, Cholis Sa'dijah, Subanji Subanji, Sudirman Sudirman</i>	347-362
	<u>LEARNING INTEGERS WITH REALISTIC MATHEMATICS EDUCATION APPROACH BASED ON ISLAMIC VALUES</u> <i>Muslimin Muslimin, Ratu Ilma Indra Putri, Zulkardi Zulkardi, Nyimas Aisyah</i>	363-384
	<u>A COMPARISON OF MATHEMATICAL TASKS TYPES USED IN INDONESIAN AND AUSTRALIAN TEXTBOOKS BASED ON GEOMETRY CONTENTS</u> <i>Miftahul Hidayah, Helen Forgasz</i>	385-404
	<u>HOW STUDENTS WORK WITH PISA-LIKE MATHEMATICAL TASKS USING COVID-19 CONTEXT</u> <i>Zulkardi Zulkardi, Meryansumayeka Meryansumayeka, Ratu Ilma Indra Putri, Zahra Alwi, Duano Sapta Nusantara, Sahala Martua Ambarita, Yulianita Maharani, Linda Puspitasari</i>	405-416
	<u>DEDUCTIVE OR INDUCTIVE? PROSPECTIVE TEACHERS' PREFERENCE OF PROOF METHOD ON AN INTERMEDIATE PROOF TASK</u> <i>Tatag Yuli Eko Siswono, Sugi Hartono, Ahmad Wachidul Kohar</i>	417-438
	<u>LEARNING GEOMETRY AND VALUES FROM PATTERNS: ETHNOMATHEMATICS ON THE BATIK PATTERNS OF YOGYAKARTA, INDONESIA</u> <i>Rully Charitas Indra Prahmana, Ubiratan D'Ambrosio</i>	439-456
	<u>CONTEXTUALIZED LEARNING MODULES IN BRIDGING STUDENTS' LEARNING GAPS IN CALCULUS WITH ANALYTIC GEOMETRY THROUGH INDEPENDENT LEARNING</u> <i>Anthony Loria Madrazo, Ryan Villareas Dio</i>	457-476
	<u>ASSESSMENT OF MATHEMATICS TEACHERS' PROFESSIONAL COMPETENCE</u> <i>Nataliya Podkhodova, Viktoria Snegurova, Natalia Stefanova, Alla Triapitsyna, Svetlana Pisareva</i>	477-500
	<u>AN ANALYSIS OF LEARNERS' SOLUTION STRATEGIES IN THE CONTEXT OF MODELLING TASKS</u> <i>Xenia-Rosemarie Reit, Marc Schäfer</i>	501-512



MATHEMATICAL CONTENT ON STEM ACTIVITIES

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Abstract

In this paper, a number of STEM educational proposals are systematically analyzed from the lens of mathematics education. An extensive innovation project was implemented during the 2019/2020 academic year in a pilot study carried out in Schools and Teacher Training Programs in Navarre (Spain), comprising a bibliographical and source analysis as a previous step to characterize the existing material, and ultimately to design and test STEM projects at different educational levels from the point of view of mathematical education. All activities belong to international publications and widely used and contrasted web repositories, and seize the usual interval of compulsory education, i.e., from the beginning of Primary School (age 6/7) to the end of Secondary School (age 15/16). The findings draw a panorama of STEM activities where mathematics is mostly utilitarian, numbers and units are functionally used to measure quantities of magnitudes, and geometric contents serve the purpose of modeling a technological prototype. As it turns out, some STEM-labelled activities do not fulfill their principles and fundamental purposes. In lower levels, there is a common confusion between STEM activities and science laboratory projects; in higher levels, complex mathematical content could appear. Even though some activities are guided science laboratory projects, it is concluded that most STEM activities have the potential of a-didactical situations, i.e., contexts where students put into practice their personal problem-solving techniques before teachers formalize the mathematical content.

Keywords: STEM, Spanish and Portuguese mathematics curriculum, Primary education, Secondary education, Didactical situations in mathematics

Abstrak

Pada makalah ini, sejumlah proposal pendidikan STEM dianalisis secara sistematis dari kacamata pendidikan matematika. Proyek inovasi ekstensif dilaksanakan selama tahun akademik 2019/2020 dalam studi percontohan yang dilakukan di Program Pelatihan Guru dan Sekolah di Navarre (Spanyol), yang terdiri dari analisis bibliografi dan sumber sebagai langkah sebelumnya untuk mengkarakterisasi materi yang ada, dan akhirnya merancang dan menguji proyek STEM di tingkat pendidikan yang berbeda dari sudut pandang pendidikan matematika. Semua aktivitas didasari oleh publikasi internasional dan repositori web yang digunakan secara luas dan kontras, dan memanfaatkan pendidikan yang wajib pada umumnya, yaitu, mulai dari awal Sekolah Dasar (usia 6/7) hingga akhir Sekolah Menengah (usia 15/16). Penemuan ini menggambarkan aktivitas STEM yang mana matematika sebagian besar bersifat *utilitarian*, bilangan dan unit secara fungsional digunakan untuk mengukur besaran, dan konten geometri berfungsi untuk memodelkan prototipe teknologi. Hasilnya, sejumlah kegiatan berlabel STEM tidak memenuhi prinsip dan tujuan yang mendasar dari konten STEM itu sendiri. Di tingkat yang lebih rendah, terdapat kesalahan umum antara aktivitas STEM dan proyek laboratorium sains; sedangkan, di tingkat yang lebih tinggi, konten matematika yang kompleks dapat muncul. Meskipun beberapa kegiatan merupakan proyek laboratorium sains terbimbing, dapat disimpulkan bahwa sebagian besar kegiatan STEM memiliki potensi situasi didaktis, sebagai contoh konteks yang mana siswa mempraktikkan teknik pemecahan masalah pribadi mereka sebelum guru memformalkan konten matematikanya.

Kata kunci: STEM, Kurikulum matematika di Spanyol dan Portugis, Pendidikan dasar, Pendidikan menengah, Situasi didaktis dalam matematika

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The sudden emergence of STEM-labelled activities in the Spanish and Portuguese educational systems around 2018 compelled *Federación Española de Sociedades de Profesores de Matemáticas* (FESPM) or Spanish Federation of Mathematic Teachers Association and *Associação de Professores de*

Matemática (APM) or Mathematics Teachers (Portuguese) Association to rethink what interdisciplinary projects mean in terms of mathematical education (Lasa, 2019). On the one hand, different mathematical content blocks in Spanish and Portuguese educational programs and curricula generally lack inner articulation and generate a growing bias towards decontextualized algebra; thus, a STEM project is an opportunity to contextualize mathematical content. On the other hand, there seems to be a lack of STEM projects where complex mathematical contents arise, making it hard to fulfil all the formal requirements of mathematics in such a project.

Lasa et al. (2020) conceived a theoretical frame to develop mathematical contents in STEM-based projects using theoretical, methodological and technological tools from the Theory of Didactical Situations in Mathematics (TDSM), the Onto-Semiotic Approach (OSA), and Instrumental Genesis (IG). In order to develop the curriculum principle (NCTM), STEM projects could be considered as didactical situations (TSDM) which contextualize algebraic contents; the formalization of the algebraic content will come later. Researchers and teachers implemented this frame within a pilot study carried out in Schools and Teacher Training Programs during the 2019/2020 academic year.

Hence, one can regard a STEM project as a scenario where students may use a number of basic strategies and apply their personal knowledge to solve a particular task. The STEM environment plays the role of the antagonistic milieu which students confront and from which they get mathematical feedback. Students take decisions, discuss different options and verify their suitability, and teachers may exploit the context to develop and shape the mathematical knowledge arising from the final project into their institutional form (Lasa, 2019).

According to the curriculum principle (NCTM, 2000), notions of proportionality (or Thales theorem in geometric context), ratio and linearity are some of the key notions that back up the mathematics curriculum in secondary school, and allow the articulation of algebra, geometry and functions. Thus, social science studies, environmental studies or STEM projects offer a setting for students to solve simple tasks using their own basic strategies (manipulation of large numbers, measurement and statistics), giving coherence to the curriculum and interlinking different mathematical ideas.

Before attempting any design on our own, it was essential to get the wider picture, i.e., the state and conditions of STEM activities in books and websites from the point of view of mathematical education. When analyzing the material, some previous hypotheses were checked, which arise from previous discussions of teachers and researchers on FESPM and APM:

- a. Mathematical content in STEM activities is basic and utilitarian, mostly related to measuring magnitudes and using fundamental geometrical language.
- b. STEM projects are seen as the final step of a learning process, where students apply new concepts they have recently learned to address a practical problem of some sort. However, this approach usually leads to a closely guided application with a lack of student initiative.

In this work, researchers intend to verify these suspicions, by systematically analyzing a number of STEM activities, on international publications and websites. For each activity, its mathematical content was examined, and whether it satisfied STEM-based suitability criteria. The results and their discussion should provide both a general view and a starting point.

Mathematics and Technology: STEM Skills and Aptitudes

The STEM approach to scientific education emphasizes an interdisciplinary educational strategy where academically rigorous concepts are coupled with real situations (Sanders, 2009), i.e., science, technology, engineering and mathematics are put into practice in contexts related to school, society, sport, or work (Tsupros et al., 2009; Berube, 2014), increasing the complexity to integrate them correctly and with coherence (Lagrange & Monaghan, 2009). STEM educational base attempts to remove barriers separating the four above-mentioned disciplines and integrate them with rigorous and meaningful learning experiences for students. Thus, researchers seek for contextualized teaching situations, where the student's environment takes on greater relevance (Laboy-Rush, 2011; Rico & Lupiañez, 2008), and such educational model should offer students tools to understand and interpret the world around them (Valencia, Méndez, & Jiménez, 2008; Hegedus et al., 2017). Briefly, science provides a context for reflection, organization and action, where technology and engineering offer technical tools enable the construction of models, and mathematics provides a set of notions and skills that allow solving problems (Lupiañez & Cruz, 2019; Lupiañez & Ruiz, 2017; Capraro, Capraro & Morgan, 2013). Therefore, students facing a STEM problem must be able to draw up an action plan for a problem, to propose solutions to the needs of today's world, and they should be creative when applying science processes, and capable of understanding their nature, using their own rational and logical initiative (Morrison, 2006; Benjumeda & Romero, 2017; Taub et al., 2018).

Source and Sample

There can be found any number of STEM activities, both in books and websites. Therefore, the focus is placed on a sample of activities meeting the following selection criteria. In the case of printed book activities (e.g., Heinecke, 2019; Hutchinson, 2019), they must be taken from largely distributed editions, with some commercial impact, translated into many languages, and widely used in different countries; in the case of websites, activities from supervised scientific repositories are considered, publicly funded and linked to verified educational projects (e.g., SCIENTIX, EU-HOU; etc.). All STEM activities are designed for Primary School students (age 7/11) and for Compulsory Secondary School students (age 12/15), and approved by FESPM and APM as high-quality educational sources.

METHOD

The research implements a multimethod methodology. Quantitative and qualitative paradigms are applied on a sequential explanatory design. The research is based on a descriptive analysis,

complemented by a documentary analysis of bibliographic review. The choice of a mixed method is justified by the statistical tools used in the analysis, since joins together statistical descriptive analysis and implicative analysis techniques (Gras, Suzuki, Guillet, & Spagnolo, 2008).

Mathematics Education in Spain and Portugal share common backgrounds (Lasa, 2019). Spanish Primary Education and Portuguese Basic Education Cycles 1 and 2 comprehend ages 7 to 11. Spanish Secondary Compulsory Education comprehends ages 12 to 15, while Portuguese Basic Education Cycle 3 comprehends ages 12 to 14. In spite of these differences, it is therefore consistent to break both levels at the common ground of age 11.

A non-intentional set of 164 STEM activities from the literature referred to in section B (source and sample), labelled with explicit mathematical content was systematically analyzed. Table 1 displays the distribution of activities according to educational levels, student ages and source supports. Half of the activities were for Primary School / Basic Cycles 1 and 2 students (7-11 years), of which 58 were taken from printed books (35.4%) and 24 from websites (14.6%). The other half were for Compulsory Secondary School / Basic Cycle 3 (12-14/15), of which 46 were taken from printed books (28.0%) and 36 were taken from websites (22.0%).

Table 1. Distribution of activities in accordance with age

Age	Source	Frequency	Percentage (%)
7-11	Books	58	35.4%
12-14/15	Books	46	28.0%
7-11	Websites	24	14.6%
12-14/15	Websites	36	22.0%
Total		164	100.0%

Characteristics and features for each activity were measured. The principal aim was to identify the mathematical content stated in the activity. Therefore, five categories were defined for mathematical content, which arise a priori from the organization of mathematical topics in school curriculum: arithmetic, measurement, statistics, geometry and functions; and researchers checked which activities stated these mathematical contents. Each activity could state one, more than one, or none. For practical reasons, arithmetic was considered a single category, as it contains algebraic content as well, secondary school being a transitional stage from one content to the other. Proportionality was also tagged as a functional content related to linearity, rather than as an independent content, due to its backbone character, i.e., a longitudinal content that backs up mathematics across the curriculum in secondary school.

Additionally, suitability factors were analyzed for each activity, i.e., researchers checked whether the activity follows objective rules in order to qualify as a STEM activity. According to Morrison (2006), students should express six abilities when facing a STEM activity, namely they should be: (1)

problem solvers, (2) innovators, (3) inventors, (4) self-sufficient, (5) logical thinkers, and (6) technologically literate. Those six abilities were gathered together into four suitability factors. Thus, it could be decided to what extent the design of a particular activity enables the emergence of these abilities, i.e., whether the activity: (a) is a challenge for the student and requires solving a problem; (b) requires innovation and inventiveness; (c) students face it with autonomy; and, (d) requires technological literacy (both digital technology and technology related to science).

For each activity, the explicit presence of a particular mathematical content was tagged by means of Boolean values: true (1) or false (0). Researchers only focused on mathematical content, even though additional scientific content should arise from the activity. They also kept qualitative track of all mathematical contents in each activity. Sometimes, a particular activity is labelled as containing mathematical content, but there are no mathematical contents at all to work with; in these cases, all values were set to zero. For each activity, they also tagged the appearance of a particular suitability factor by means of Boolean values: true (1) or false (0).

Researchers compiled data from the sample activities, and analyzed these data using two techniques. First, they made a statistical descriptive analysis: they measured frequencies and percentages to make a quantitative description of the activities. Then, they analyzed the sample using implicative analysis techniques (Gras, Suzuki, Guillet, & Spagnolo, 2008). This second approach could ensure the conclusions from the descriptive analysis, and could lead to hidden relations between variables, which are hard to see at the descriptive level.

RESULTS AND DISCUSSION

The 34 activities in this sample (20.7%) do not envisage any mathematical content at all. 28 of these 34 activities are taken from books and addressed to primary school students, i.e., half of the activities in primary school books do not envisage any formal use of mathematical contents. The lack of mathematical contents appears, mostly, in artistic activities (handmade production processes, e.g., paper, sculptures; activities with natural object, e.g., fruits and vegetables), and science activities (experimental introduction to notions of physics and natural sciences, e.g., pressure, light, solar energy, ecological wrappers). Engineering and technological projects do envisage a more mathematical activity. However, sometimes the aim of the activity is to illustrate the functioning of technological prototypes, so mathematical content also disappears (e.g., airplanes, parachutes, drones, bridges, internet, telephone, satellites, spin-dryers, zippers).

There are 130 activities (79.3%) using contents from at least one mathematical category. Table 2 displays mathematical content frequencies for all activities; percentages do not add up to 100%, since more than one mathematical content could appear in a particular task. Note that measurement and geometrical contents appear most frequently, followed by arithmetic, statistics and functions.

Table 2. Mathematical categories

Category	Frequency	Percentage (%)
Measurement	80	48.8
Geometry	49	29.8
Arithmetic	33	20.1
Statistics	25	15.2
Functions	20	12.2

On the one hand, activities tagged as engineering or technology mostly deal with measurement and geometry contents. On the other hand, activities tagged as science or art do not have much mathematical content. Finally, there are specific activities tagged as mathematics, where complex geometric and numerical notions arise.

When facing tasks tagged as engineering or technology (46 activities have this label), students are required to measure lengths, mostly in centimeters. Engineering tasks have a number of geometrical contents for: modelling (e.g., square, line, parallel, horizontal, vertical, diagonal, middle point; circumference, circle, arc, angle; lens, projection; direction, orientation, axis of rotation, mirror, reflection; symmetry, rotation); using geometrical drawing instruments (rule and compass); fundamental function contents (e.g., vibration, wave); basic probability and random notions; and search for patterns. In technology activities, students mostly measure the intensity of electric currents (A), and they use positive and negative numbers to do so. The measurement of lengths (m, cm) and capacities (ml) is required when dealing with mechanical prototypes, and geometric language (e.g., circumference, center, diameter, line, perpendicular, middle point, cut point; axis of rotation, angle, direction, orientation) and drawing instruments (rule and compass) arise in this context, as well as random trajectories and algorithms.

Engineering and technology STEM projects deal with workshop construction of many instruments, such as: telescopes (Thales Theorem and the measurement of distances); flashlights (electrical magnitudes); turbines and pendulums (rotation, oscillation); and kaleidoscopes (symmetry, reflexion-angle). Sometimes, students must build and program a robot in order to accomplish a particular task, using software and sensors: trajectory modelling, speed, etc.

In mathematic-tagged activities (24 labelled activities), measurement does appear too. Students are required to deal with a variety of magnitudes, and they deepen in the measurement process itself. They estimate results before making any calculation, compare approximate and precise measures using symbolic representations ($\approx$, $=$), convert measures from non-standard to standard units and from one unit to another, and measure accurate units other than centimeters (mm), as well as many other magnitudes (e.g., capacity, ml, l; mass, g, kg; temperature, C; time, h, min). Geometric contents in mathematic-tagged activities deal with elementary plane geometry (e.g., square, triangle, similarity; circumference, arc, line, curve; edge, middle point, median), and geometrical drawing instruments (rule

and compass), but they also release complex mathematical contents (e.g., fractal geometry, snowflake geometry, mosaic and tessellation; spiral, Fibonacci spiral, loops, Moebius band, projective geometry and shadows). GeoGebra usually supports geometrical activities, and modelling usually requires 3D printers (Nisiyatussani et al., 2018).

In activities tagged as science (23), students mostly measure capacity (ml) and time (day, min), as well as mass (g) and temperature (C). These activities deal with notions from natural sciences and physics, and basic mathematical contents arise, such as the use of positive and negative numbers, statistical variables, and basic geometric notions. For example, students may collect data from the Solar System, in order to define its rotational model; deal with force and mass to balance weights; etc.

Only 2 artistic activities in this sample contain measurement notions that have to do with mass (g), capacity (ml, l), and time (s). Table 3 summarizes the use of magnitudes and their measurements in sample activities.

Table 3. Magnitudes in measurement activities

Magnitude	Units	Frequency	%
Length	<i>m, cm, mm</i>	41	51.3
Volume, capacity	<i>ml, l</i>	21	26.3
Time	<i>Day, h, min, s</i>	15	18.8
Mass	<i>Kg, g</i>	10	12.5
Area	<i>cm²</i>	7	8.8
Temperature	<i>°C, °F</i>	7	8.8
Money	<i>€, \$</i>	7	8.8
Amplitude	<i>°, ', "</i>	7	8.8
Electric intensity	<i>A</i>	5	6.3
Light intensity	<i>Candle</i>	3	3.8
Energy	<i>J</i>	2	2.5
Density	<i>Kg × m⁻³</i>	2	2.5
Power	<i>N</i>	1	1.3
Wavelength	<i>Hertz</i>	1	1.3
Speed	<i>m/s, Km/h</i>	1	1.3

11 activities from the sample (out of 25; Table 2) are based on statistical content. Students are taught about data collection, data analysis and data representation techniques (e.g., double input table, tree-diagram, bar-diagram), using spreadsheets. Then, students use data with a specific purpose (e.g., geometrical design of an interior space, or a school; geometrical modelling and 3D printing of a beehive, or a bird-nest; measurement, quantification, and classification of waste in a recycling factory; pulse steadiness graphs; time-laps and speed measurement of vehicles in various routes). 6 statistical activities

for young primary school pupils deal with the identification of sensorial qualitative variables (stiff/flexible; robust/weak; rough/smooth; seriation of sensorial characters).

7 activities from the sample (out 33; [Table 2](#)) are based on arithmetical content. The purpose of these activities is to show young pupils fundamental arithmetical notions: counting and cardinals; the difference between recursive and Cartesian multiplication; division; use of positive and negative numbers; rational numbers; rounding up and down; etc. Arithmetic notions arise in mathematic-tagged activities, dealing with: fractions and powers: Fibonacci sequence: Eratosthenes method: second grade equations: etc. An extract from the history of mathematics provides the background to introduce complex arithmetic and geometry contents.

Activities in other groups or classifications can be found. For instance, 10 activities in the sample deal with optimization problems. In such activities, students design geometrical or functional models according to statistical data (e.g., spaces and volumes, construction or insulating material costs, cloth measures depending on growth rates, insurance products), and build physical prototypes (e.g., scale models).

In 107 activities (65.2%), none of the four suitability factors are identified. All book activities are included in this category. In 57 activities (34.8%), at least one suitability factor can be identified. Most website activities fulfil one or more than one suitability factors ([Table 4](#)). 54 activities are considered to be (a) a challenge for students and require to solve a problem; 32 activities (b) require innovation and inventiveness; 57 activities (c) are faced by students with autonomy; and 38 activities (d) require technological literacy. There is no significant difference between requirements on the use of digital technology (55.3%) and science-related technology (44.7%). 18 activities fulfil all four suitability factors, and 32 fulfil three of them. All 3-SF activities are (a) a challenge for students and require to solve a problem, and (c) students face them with autonomy.

Table 4. Suitability factors

SF	Fr.	%
(a)	54	32.9
(b)	32	19.5
(c)	57	34.8
(d)	38	22.6
All 4	18	11.0
3 out of 4	32	19.5
2 out of 4	5	3.0
1 out of 4	2	1.2

On the one hand, only one single activity (no. 123) in the sample could be identified where all mathematical contents are considered, and which fulfil all four suitability factors. In this activity,

students design and build a water rocket, and they must launch it as high as possible (<https://www.stem4math.eu/mission-mars>). Instructions deal with real world motivation: “In the 21st century, many countries and space agencies are attempting to send probes and landers to Mars. Mars is an interesting planet because conditions on its surface may have been very similar to conditions on Earth. The ESA (European Space Agency) sent a lander called Schiaparelli to Mars at the beginning of 2016. In October 2016, Schiaparelli reached Mars, but something went wrong with its landing. Unfortunately, the lander was completely destroyed”.

In activity no. 123, students must investigate the influence of different variables on the flight of the rocket (*science*), designing and constructing the best possible water rocket (*technology* and *engineering*). *Mathematical* skills take into account scaling, calculating averages, calculating big numbers (to stick to a budget), measuring volumes of water, and working on proportions. Suitability factors include: asking questions and problem-solving (e.g., controlling variables in order to find the best possible rocket design); planning and budgeting; collecting, analyzing and interpreting data (e.g., why is it important to measure something several times?); reporting data (e.g., explaining what influence different variables had on rocket flight); and reflecting (e.g., what process did we go through in finding and producing the best possible rocket?).

On the other hand, 30 activities in the sample were identified where no mathematical contents are considered, and which fulfil none of the suitability factors (out of 34 activities, where no mathematical contents are considered), e.g., (no. 131) students design a healthy, multicultural meal for their school by investigating and collaborating (<http://www.scientix.eu/resources/details?resourceId=23080>). Activity no. 131 is useful and educational, but cannot be included in a STEM catalogue:

“In this open activity students design a healthy, multicultural meal for their school by investigating and collaborating. How healthy a meal is, is described in relation to the five food groups and a calculation of the nutritional value. Multiculturalism is explored by investigating the students' eating habits and linking them to their cultural backgrounds. Collaborating on an open and rich task allows for diversity in the ways of working as well as in the level of achievement. Students can discuss and support one another and/or divide tasks according to their preference, ability, etc.”

Sometimes, the main purpose of a STEM activity is to illustrate a natural principle or give an example of a technological prototype. These activities rarely show any mathematical activity. Electronics based projects or user-level software-based projects are clear examples. This also happens when an activity uses a scientific principle for artistic purposes (e.g., music or sculpture). Even in contexts where the use of numerical models could be justified and would be interesting (e.g., estimate how much food can the Earth produce; design of nesting nets), this opportunity is sometimes not fully exploited in the activities.

A glance at Table 2 reveals the utilitarian use of mathematics in STEM activities. In most tasks, numbers and units are helpful to measure lengths for technological prototypes, and they are useful to

measure volumes in laboratory experiments, i.e., mathematics is functional to measure magnitude quantities. Geometric contents are utilitarian too, since they serve the purpose of modelling a technological prototype, and they target the design of parts, pieces and elements of a mechanism (Apsari et al., 2020). Sometimes, a STEM project is tagged as *mathematical*, and the educational objectives build on those same notions relating to magnitudes and geometric language; yet, they rarely introduce any other complex mathematical content.

Mostly at lower education levels, STEM projects and science laboratory projects are jumbled together, and both terms are indistinguishable in the same contexts. Anyhow, STEM activities demonstrate that, even at lower levels, pupils can face STEM projects, using numbers and statistical notions to tackle actual problems, e.g., young pupils can analyze physical phenomena and sensorial variables by using qualitative perception and numbers.

It is hard to find reliable STEM activities requiring to solve an interesting problem with inventiveness, autonomy, and technological literacy. Findings shows that books are not a good source of STEM activities anymore: numerical data, dynamic models, additional information, etc., are more easily found in websites and links; and objective analysis shows that website proposals are more satisfactory from the point of view of task-design.

Statistical implicative analysis confirms previous assertions. Table 5 summarizes meanings and values for one external variable (source support), five internal variables with mathematical content information, and four internal variables with suitability factors information.

Table 5. Variables and implicative analysis

Variable	Meaning and value
IT	Source: Website (1), Book (0)
MC1	Arithmetical content: Yes (1), No (0)
MC2	Measurement content: Yes (1), No (0)
MC3	Statistical content: Yes (1), No (0)
MC4	Geometrical content: Yes (1), No (0)
MC5	Functional content: Yes (1), No (0)
SF1	Challenge problem: Yes (1), No (0)
SF2	Innovation inventiveness: Yes (1), No (0)
SF3	Students autonomy: Yes (1), No (0)
SF4	Technological literacy: Yes (1), No (0)

Implicative analysis confirms the strong relationship between activity support and suitability factors. The implicative tree (Figure 1) gathers IT and SF variables together in the main branch, i.e., these variables contribute most to the main branch, in terms of frequency. Measurement and statistical mathematical contents make an important contribution to the main branch too. Some other relationships

arise which were hard to see with descriptive statistical procedures. For instance, arithmetical and functional contents share a strong similitude index, i.e., STEM activities envisage the use of numbers and operations, along with functions and proportionality, and vice versa. Geometrical contents constitute a diversity of backgrounds for many activities, with weak relationship to just one other variable.

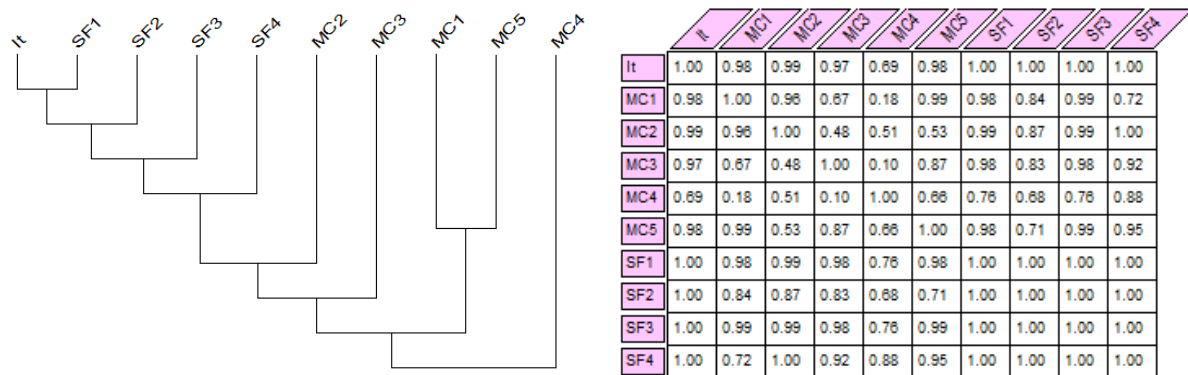


Figure 1. Implicative tree and similitude index

The implicative graph (Figure 2) shows the actual direction of the implications. Red arrows show a 99% implicative relationship between variables. There is a 99% implicative relationship from variables SF4 / SF2 to SF1, meaning that activities requiring technological literacy, as well as activities requiring innovation and inventiveness, are a challenge for students when they require problem-solving skills. There is a 99% implicative relationship from SF1 to SF3 (challenge activities imply student face it with autonomy) and from SF3 to IT (autonomous activities found in websites): this section of the implicative chain shows strong correlation between variables (Figure 2). Finally, there is a 99% implicative relationship from IT to MC2 (website activities dealing with magnitudes and measurements).

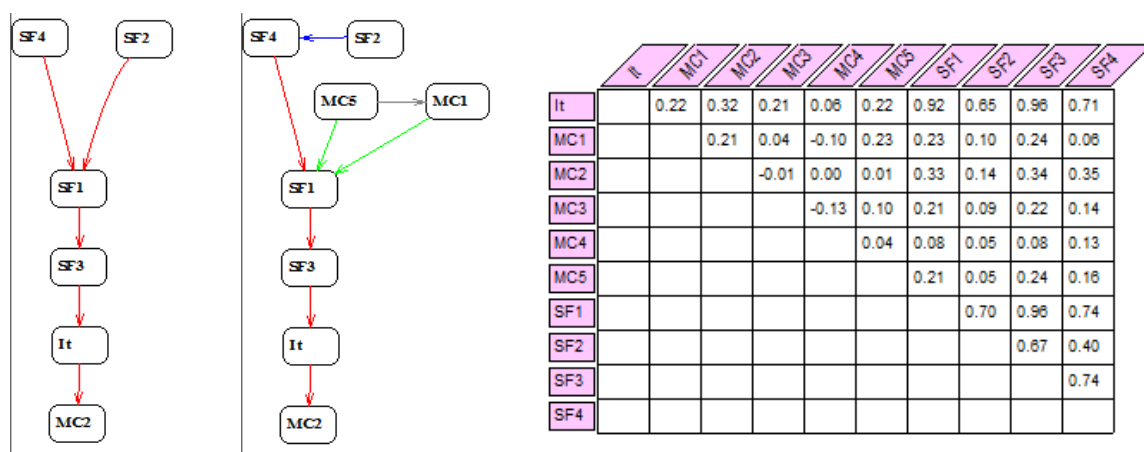


Figure 2. Implicative graphs and correlations

At a lower level, the blue arrow shows a 95% implicative relationship between variables: activities requiring innovation and inventiveness also require technological literacy. Green (90%) and grey (85%) arrows show weak implicative relationships between variables. The use of functional and arithmetical contents implies a challenge or a problem-solving context, and functional mathematical contents imply the use of arithmetical (algebraic) language.

Teachers and researchers must be aware on the inner articulation of mathematical content in STEM proposals, in order to design faithful learning situations from the mathematical point of view. Sharing this knowledge within the community will lessen fear and afford teachers confidence (Chahine, Robinson, & Mansion, 2020). On the other hand, the findings assert the point of view that STEM projects hold the potential to be used as a-didactical situations.

CONCLUSION

Asserting hypothesis (A), mathematical content in STEM activities is basic and utilitarian, being mostly related to the measurement of magnitudes, statistical interpretation of numbers and the use of fundamental geometric language. Since students do not require complex knowledge of mathematics to address a STEM activity, the activity itself is a potential a-didactical situation. Suitable activities reinforce challenge, inventiveness, autonomy and technological literacy, all elements related to a didactical situation. This approach would be used in the future to integrate the different areas of knowledge in interdisciplinary processes, avoiding asymmetry of roles, and testing constraints and obstacles when implementing interdisciplinary and STEM activities.

Thus, instead of considering the STEM project as the final step of a learning process, where students apply new concepts they have recently learned to address a practical problem of some sort, it could be considered as a context where students put into practice their personal problem-solving skills. Afterwards, the teacher could make use of this resort to formalize the arithmetical (algebraic) and functional contents arising from the activity into their institutional meaning. The first approach usually leads to a closely guided application, with a lack of students' initiative, and could cause Topaz effects (Brousseau, 1997).

STEM is the acronym for four disciplines, aiming to bring Science, Technology, Engineering AND Mathematics together. All four disciplines should arise at the same time. A fashionable use of STEM shows that in many activities just one of the four disciplines is considered, thus replacing the conjunction AND with OR. However, this should not be our style.

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CHARACTERISTICS OF STUDENTS' ABDUCTIVE REASONING IN SOLVING ALGEBRA PROBLEMS

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Abstract

When students solve an algebra problem, students try to deduce the facts in the problem. This step is imperative, students can draw conclusions from the facts and devise a plan to solve the problem. Drawing conclusions from facts is called reasoning. Some kinds of reasoning are deductive, inductive, and abductive. This article explores the characteristics of some types of abductive reasoning used by mathematics education students in problem-solving related to using facts on the problems. Fifty-eight students were asked to solve an algebra problem. It was found that the student's solutions could be grouped into four types of abductive reasoning. From each group, one student was interviewed to have more details on the types. First, the creative conjectures type, the students can solve the problems and develop new ideas related to the problems; second, fact optimization type, the students make conjecture of the answer, then confirm it by deductive reasoning; third, factual error type, students use facts outside of the problems to solve it, but the facts are wrong; and fourth, mistaken fact type, the students assume the questionable thing as a given fact. Therefore, teachers should encourage the students to use creative conjectures and fact optimization when learning mathematics.

Keywords: Characteristics, Abductive reasoning, Algebra problems

Abstrak

Ketika mahasiswa menyelesaikan masalah aljabar, mahasiswa berusaha untuk menyimpulkan fakta-fakta yang terdapat pada masalah. Langkah ini sangat penting karena mahasiswa dapat menarik kesimpulan dari fakta-fakta dan merancang rencana untuk menyelesaikan masalah. Penarikan kesimpulan dari fakta-fakta disebut penalaran. Beberapa jenis penalaran di antaranya adalah penalaran deduktif, induktif dan abduktif. Artikel ini mendiskusikan karakteristik dari beberapa tipe penalaran abduktif yang digunakan oleh mahasiswa pendidikan matematika dalam penyelesaian masalah yang dikaitkan dengan penggunaan fakta pada masalah. Lima puluh delapan mahasiswa diminta untuk menyelesaikan masalah aljabar. Ditemukan bahwa jawaban mahasiswa dapat dikelompokkan ke dalam empat tipe penalaran abduktif. Dari masing-masing kelompok, diambil satu mahasiswa untuk diwawancara dengan tujuan mendapatkan penjelasan yang lebih detil dari masing-masing tipe tersebut. Empat tipe tersebut dapat dijelaskan sebagai berikut, pertama, tipe kreatif konjektur, mahasiswa dapat menyelesaikan masalah dan mengembangkan ide baru yang berkaitan dengan masalah; kedua, tipe optimalisasi fakta, mahasiswa membuat konjektur tentang jawaban masalah, kemudian mengonfirmasi konjektur dengan melakukan penalaran deduktif; ketiga, tipe fakta salah, mahasiswa mengambil fakta di luar soal untuk menyelesaikan masalah, namun fakta yang diambil salah; dan keempat, tipe kesalahan fakta, mahasiswa menganggap hal yang ditanyakan pada soal sebagai fakta yang diberikan. Oleh karena itu, dosen seharusnya mendorong mahasiswa untuk menggunakan kreatif konjektur dan optimalisasi fakta ketika mempelajari matematika.

Kata kunci: Karakteristik, Penalaran abduktif, Masalah aljabar

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Problem-solving has been regarded as an essential domain in the teaching and learning of mathematics, though it takes several requirements to understand the process of doing problem-solving. In recent years, several studies have been carried out to explore the essence of problem-solving in mathematics learning (Gurat, 2018; Jäder et al., 2019; Chew et al., 2019; Reiss & Törner, 2007; Ekawati et al., 2019). For instance, Gurat (2018) discussed problem-solving strategies in teacher-student interaction, and

Reiss and Törner (2007) investigated cognitive psychologists influencing students to solve mathematics problems in a classroom. A recent report also informed that problem-solving learning must be supported by the availability of books (Jäder et al., 2019).

Based on our classroom observations, some students encountered difficulties in problem-solving activities. Their inability may be caused by the complexities of making conclusions of the given facts in the problem-solving. Students' failure in problem-solving shows that the importance of problem-solving has not been well-taught in mathematics learning.

One of the prominent factors that can support problem-solving in practice is reasoning. Therefore, NCTM (2000) emphasizes the importance of reasoning and evidence as fundamental aspects of mathematics learning. Theoretically, the reasoning is defined as the process of drawing conclusions (Leighton & Sternberg, 2003; Sternberg & Sternberg, 2012). Reasoning and evidence are needed in building up a reasonable argument to prove the truth of a statement.

Reasoning in mathematics learning has been found to be varied such as quantitative reasoning (Moore, 2014), covariational reasoning (Subanji & Supratman, 2015), proportional reasoning (Im & Jitendra, 2020), analogy reasoning (Lailiyah, et al., 2018), algebra reasoning (Otten et al., 2019), and many more. The nature of mathematical thinking serves as a character in reasoning. In this research, one aspect of thinking in mathematics learning that is drawing a conclusion from an abductive perspective is discussed. In particular, the term abductive reasoning is used in this paper.

Abductive reasoning is closely related to human lives in the form of conjecture that is made based on some facts. Abductive reasoning is also widely explored in several fields, such as in medical diagnosis (Alirezaie & Loutfi, 2014; Aliseda, 2006), computer and programing (Dong et al., 2015; Ma et al., 2008), and critical thinking-related activities (O'Reilly, 2016).

In the context of mathematics learning, abductive reasoning is adopted to improve discovery making and develop students' creative reasoning (Ferguson, 2019). Other researchers have developed a learning model by using abductive reasoning and drawing the conclusion process to generate conjectures in geometry (Baccaglini-Frank, 2019), explore the effect of learning with abductive-deductive reasoning (Shodikin, 2017), and facilitate students' inquiries in calculus (Park & Lee, 2016). The importance of abductive reasoning in problem-solving is discussed further by Cifarelli (2016).

Abductive reasoning is another type of reasoning, besides deductive and inductive reasoning (Peirce, 1958). Deductive reasoning is defined as the process of reasoning from general principles to particular cases (Cohen & Stemmer, 2007). It involves reasoning from one or more general statements that are known to a specific case of the general statement. In contrast, deductive reasoning involves reaching conclusions from a set of conditional propositions or a syllogistic pair of premises (Sternberg & Sternberg, 2012).

Inductive reasoning is the process of reasoning from specific facts or observations to reach a conclusion that may explain the facts (Johnson-Laird, 1999). Inductive reasoning is a type of reasoning which takes a conclusion as approximate as possible because it is derived from a method of inference

that generally leads to the truth in the long run (Peirce, 1958). Although we cannot reach logically certain conclusions through inductive reasoning, we can still at least reach highly probable conclusions through careful reasoning (Sternberg & Sternberg, 2012).

Abductive reasoning is defined as the process of forming an explanatory hypothesis from an observation that requires an explanation. Abductive reasoning is started from the facts, uses any particular theory in view, and it is motivated by the feeling that a theory is needed to explain the surprising facts. Abductive reasoning is a data-driven process and also dependent on the knowledge domain. This process is not algorithmic, which is less on its logical form and an act of insight, although it is extremely fallible insight (Peirce, 1958). Abductive reasoning is often portrayed as explanatory reasoning that leads back from facts to a proposed explanation of those facts (Seel, 2012).

Deductive, inductive, and abductive reasoning plays an important role together in scientific inquiry to find new knowledge (Bellucci, 2018; Moscoso & Palacios, 2019; Peirce, 1958; Shodikin, 2017). Abduction plays a role as the track from facts towards ideas and theories, while induction is the path from concepts and theories towards facts, and deductive plays a role to prove a theory.

Students can form inferences using abductive reasoning when given facts in the problem-solving activities. The inference is indeed unnecessary and logically correct, so it needs to be investigated for its truth by using deductive reasoning. According to some experts, abductive reasoning proposes a plausible conclusion, inductive reasoning proposes a probable conclusion, and deductive reasoning produces a certain conclusion (Hwang et al., 2019; Mirabile & Douven, 2020; Moscoso & Palacios, 2019; Peirce, 1958). From this explanation, it appears that what plays a role in the forming of knowledge, ideas, or new ideas is abductive reasoning. We need abductive reasoning if we wish to "learn anything" new (Niiniluoto, 2018), and abductive reasoning can produce a piece of new knowledge (Rapanta, 2018).

We will understand abductive reasoning through an example of deductive, inductive, and abductive reasoning. The following is an example of deductive, inductive, and abductive reasoning made by Peirce (1958). The examples contain some facts and inferences. The example of deductive reasoning is: *"Suppose there is a bag that only contains red marbles, and you take one (as facts). You can conclude that the marble is red."* From the example, *"there is a bag that only contains red marbles"* is a general fact, and *"we take one marble"* is a fact. The conclusion, that is, *"the marble has taken is red"*, is a particular case. In the example, we conclude by modus ponens rule. Thus, the drawing conclusion is valid by deductive reasoning. We can use modus tollens, syllogism or another rule to draw conclusions provided such that the argument is valid. The following is an example of inductive reasoning: *"Suppose there is a bag on the table, but you don't know the color of the marbles in the bag. You take one by one marble, and it's red. You may infer that all the marbles in the bag are red."* From the example of inductive reasoning, the proposition, *"We take one marble from a bag"* is a fact, and *"It's red"* is a specific fact. If we take one by one marble from a bag, although we do not take all

marbles in the bag, we can conclude that all marbles in the bag are red. Even though the drawing conclusion by inductive is not mathematically valid but sometimes inductive reasoning used in science.

The example of abductive reasoning is: “Suppose there is a bag on the table. All marbles in the bag are red, and you find a red marble around a bag. You can conclude that the red marble is from the bag.” From the example above, “All marbles in the bag are red”, and “you find a red marble around a bag” is some facts. The conclusion “the red marble is from the bag” could serve as part of an explanation for the fact. The conclusion “the red marble is from the bag” may be true or false. We must find explanations so that we get the conclusion which true. Finding explaining hypotheses is one of the most challenging creative tasks (Preyer & Mans, 1999).

Thus, we can draw a conclusion based on some facts. In problem-solving, some facts can be founded from the problems. If people who do a reason think that something is true, then something is called a fact (Ferrando, 2006). It means that a matter is considered a fact or not depends on each individual as a problem solver. Some students can find a fact easily, but others cannot. Therefore, it is fascinating to explore the students’ abductive reasoning in making conclusions based on the use of the fact.

Fact is a piece of information that can be found in a problem. Something is called fact if people think that something is true. From some facts, we can draw a conclusion. By abductive reasoning, the conclusion is stated as a conjecture. A conjecture is a logical statement, but its truth has not been confirmed (Cañadas & Castro, 2007). A conjecture can play a role as a "hypothesis," which is an idea suggested as a possible explanation for a particular situation or condition and as a "fact-conjecture" that is the final answer to a problem or solution to specific steps of the settlement process. Facts and conjectures can be expressed in the statement form (Ferrando, 2006). In a problem-solving process, we can make a conjecture many times.

Conclusions by using deductive reasoning are valid because their arguments form a tautology. In contrast, abductive reasoning produces findings that are 'not always true' so that further clarification is needed to check the truth of the conclusions. Therefore, abductive reasoning can arise an opportunity to explore new ideas that are indispensable for the development of knowledge in overcoming problems. One exciting part of abductive reasoning is it can produce new knowledge for someone who has a reason. The new knowledge in mathematics can be in the form of a new concept or unique nature (Gonzalez & Haselager, 2005; Niiniluoto, 2018).

Several studies have been carried out to investigate abductive reasoning enacted by students in high school algebra contexts. These studies have informed that generalization processes figural patterns in algebra (Rivera, 2013; Rivera, 2010; Rivera & Becker, 2007). While research of abductive reasoning in high school contexts has been widely explored, little attention has been directed to uncovering abductive reasoning enacted by university students majoring in mathematics education. Thereby, to fill such a void, the present study was designed to explore some types of abductive reasoning employed by mathematics education students in higher education contexts to solve algebra problems related to using

facts on the problems. During the enactment of this study, the students were taking algebra courses. Some of the topics were Group Theory, as in Gallian (2016). Thus, this study was set specifically on algebra for undergraduate students, which discerned the Group Theory.

METHOD

The participants in this study were 58 second-year students of a public university based in Malang, East Java, Indonesia. On 12 March 2020, they were asked to solve algebra problems. Their solutions were included in four groups based on abductive reasoning of using facts. Furthermore, to explore the characteristics of each group, we carried out task-based interviews with one student from each group. Thus, there were four students to be interviewed. They were chosen because (1) they fulfill the indicators of abductive reasoning of using facts, (2) they have good communication skills, and (3) they are willing to participate in the study.

To observe the process of students' abductive reasoning and record the interviews, an observation sheet was employed. It consists of one open-ended question that allows students to do reasoning in the problem-solving process. The instruments have been tested for validity and reliability before it was used. Content validation of questions and interview sheets was conducted by two mathematics experts and two education experts. The item of the validity of the instrument included the eligibility of the item test, the truth of concept, multiple interpretations, and appropriate instructions to do abductive reasoning. The algebra problems provided several facts, and the participants were asked to check whether a property is true or not based on the fact. The problem can be seen in Figure 1.

Suppose $\mathbb{Z}_8$ is a set of the integers modulo 8 under multiplication.
If $\forall a, b, c \in \mathbb{Z}_8, ac = bc$, is $a = b$? Please give some reasons for your answer

Figure 1. Algebra Problem to Observe Students' Abductive Reasoning

To analyze the data, first, we administered a problem to the students, and we sorted it based on participants' reasoning when doing the problem-solving process. Their solution can be grouped into four groups based on abductive reasoning related to using facts. From each group, one student was observed in-depth related to the reasoning aspect. The triangulation process was done to verify the data collected from the interview. The triangulation was also carried out to confirm the finding from the students' answers. We code students as S and researchers as R. Finally, we summarized abductive reasoning from four students in solving the problem.

RESULTS AND DISCUSSION

From four groups with different kinds of answers, the first group involves one student who can bring up new conjecture beyond the questions related to the problems. The second group that involves twenty students use facts that are "true" outside the problem to solve the problem. The third group that

involves twelve students, use facts that are “false” outside the problem to solve the problem. The fourth group that involves twenty-five students assume the questionable thing as a given fact. To know deeply how students’ reasoning is, the following are the results of the interview with the four students, namely Student 1 (S1), Student 2 (S2), Student 3 (S3), and Student 4 (S4).

Abductive Reasoning in Problem Solving Process of S1

S1 has written down all the facts in the problem. It means that S1 understands what is given and what has to be found in the problem. From the student sheet, it was found that S1 also wrote, as seen in Figure 2.

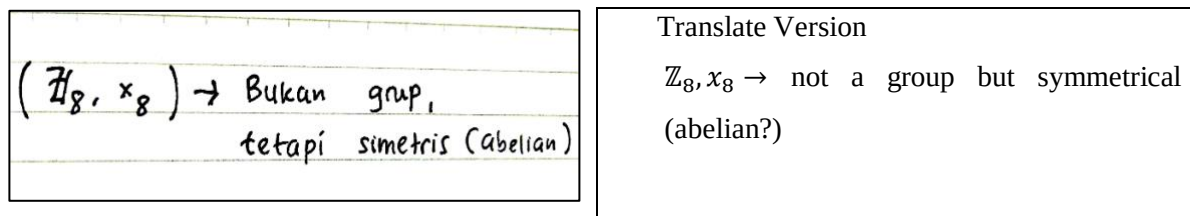


Figure 2. Conjecture by S1

From the results of the interview with S1, she made the conjecture that built based on Table Cayley made by S1. Following interviews with S1.

- R : Why do you write question mark “?” in a sentence “ $\mathbb{Z}_8, \times_8 \rightarrow \text{not a group but symmetrical (abelian?)}$ ”
- S1 : Because I am not sure about my answer, ma'am
- R : What part are you not sure?
- S1 : On the symmetric (she means abelian)

S1 makes a conjecture that “ $\mathbb{Z}_8$ is not a group under multiplication” but she still doubts the commutative property under multiplication. S1 arranges a table of a multiplication modulo 8 to persuade her conjecture. So, by abductive reasoning, S1 makes a conjecture and explains it based Cayley Table (Thagard & Shelley, 1997). Although a commutative property in $\mathbb{Z}_8$ is not used in problem-solving, but a commutative property is appeared as part of the abductive reasoning process to find new ideas of S1.

The action taken by S1 by making a multiplication table to convince the conjecture is appropriate that the activity of deciding conclusions obtained from feeling, seeing, and hearing, is useful when used to build a conjecture (Farah, 1988; Ferrando, 2006) and pictures (in this case the modulo 8 multiplication table) can be made in a creative way to explain the conjecture (Finke & Slayton, 1988). The making conclusion by abductive reasoning of S1 can be described as follows.

(Fact) The set $\mathbb{Z}_8$ with multiplication modulo 8

multiplication table modulo 8 of $\mathbb{Z}_8$

$\therefore \mathbb{Z}_8$ is not a group but $\mathbb{Z}_8$ is symmetric (it's mean abelian)

The fact, "multiplication modulo 8 table of $\mathbb{Z}_8$ " is not clearly stated in the question, but S1 raised it to help to solve algebra problems. The conjecture stated in the statement " $\mathbb{Z}_8$ is not a group but symmetrical (abelian?)" is a temporary conclusion that decided from the facts. It means that at that time, S1 made a conclusion that was not based on deductive or inductive reasoning. The conclusion made by S1 is something new. We can see that abductive reasoning can build a new idea (Eco, 1984; Moscoso & Palacios, 2019; Niiniluoto, 2018; Peirce, 1958). The next step taken by S1 is writing the statement as shown in Figure 3.

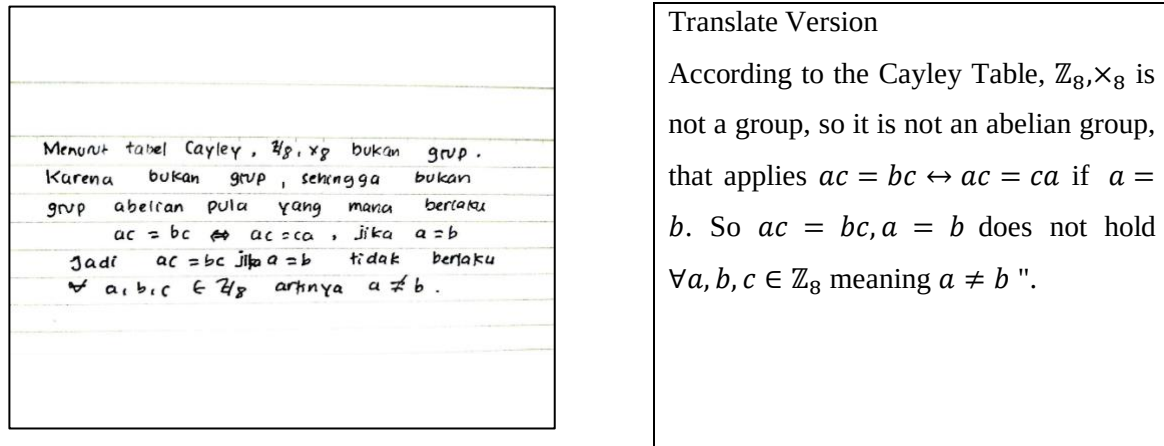


Figure 3. Abductive Statement and Its Explanation

S1 concluded that " $\forall a, b, c \in \mathbb{Z}_8, ac = bc$ but $a \neq b$ " based on the Cayley Table. So actually S1 understands that to show that the statement is false, she must make one counterexample such as $a = 4$ and $c = 3, ac = 4$. From the interviews, it was seen that to get $b \in \mathbb{Z}_8$, S1 used trial and error as seen in the following interview.

- R : Why are a, b, c in $\mathbb{Z}_8$? try to check again, is the sentence "If $ac = bc$ for any $a, b, c \in \mathbb{Z}_8$, then $a = b$ " true?
- S1 : I was trying like this, can I, ma'am? I take $a = 4$ and $c = 3, ac = 4$,
 Because $ac = bc, bc = 4$. Because $c = 3$, then the possible is $b = 4$. So $a = b$

The drawing conclusion by abductive reasoning of S1 can be described as follows.

(Fact) $(\mathbb{Z}_8, \times_8)$ not a group $\rightarrow (\mathbb{Z}_8, \times_8)$ is not an abelian group anyway

$$ac = bc \Leftrightarrow ac = ca \text{ jika } a = b.$$

$\therefore ac = bc \rightarrow a = b$ is not held.

The drawing conclusion made by S1 is not deductive nor inductive. However, the conclusion decided is true that " $ac = bc \rightarrow a = b$ is not held". It should be $\exists a, b, c \in \mathbb{Z}_8$ with $ac = bc$ but $a \neq b$.

From this step, the problem should have been answered even though the answer to S1 was not complete because there was still a slight error, but in fact, S1 continued the answer also though the

problem did not ask it. S1 had a desire to find another set with the multiplication of modulo 8 so that it applies " $ac = bc \rightarrow a = b$ ". The statement that appears is "if you discard some elements of $\mathbb{Z}_8$ that causes $\mathbb{Z}_8$ is failed to become a group then a new set, that is $U(8)$ will become group under multiplication modulo 8". A set $U(8)$ is a set of all positive integers less than 8 and relatively prime to 8. So, proposition $\forall a, b, c \in U(8), ac = bc \rightarrow a = b$ is true, as shown in Figure 4.

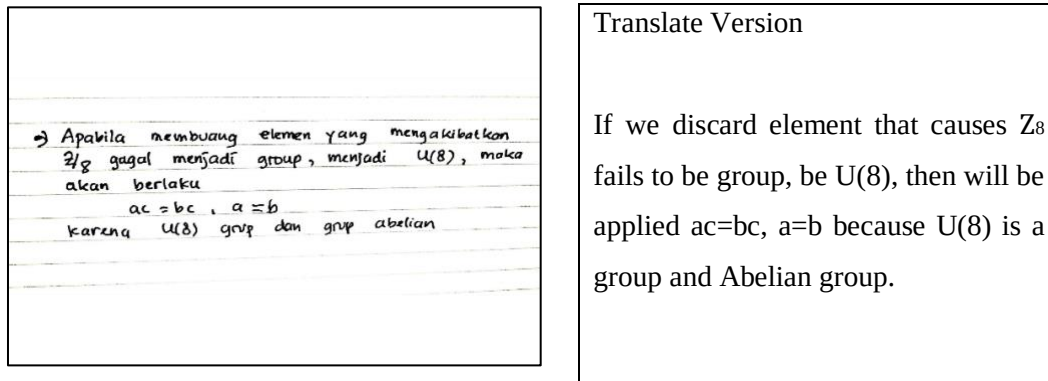


Figure 4. New Ideas by S1

S1 finds new ideas, conception, and knowledge. This is consistent with Niiniluoto (2018) that abductive reasoning is done when we are learning new things (Niiniluoto, 2018). It was also stated that abductive reasoning is bringing up new ideas that are tentative and related to the context being discussed (Duarte, 2019; O'Reilly, 2016; Tomiyama et al., 2010).

Abductive Reasoning in Problem Solving Process of S2

S2 can solve problems very precisely. They are using all the facts inside the problems and knowing that asking about it. The student uses facts that "true" outside the problems to solve the problems, that is, a set $\mathbb{Z}_8$ is not group under multiplication modulo 8. Students in this group made appropriate conjectures. The conjecture is " $\forall a, b, c \in \mathbb{Z}_8$ if $ac = bc$, then $a = b$ is false." The action taken by students of this type is finding out " $\forall a, b, c \in \mathbb{Z}_8$ who fulfills $ac = bc$, but $a \neq b$ ". This action is very appropriate for solving problems.

Abductive Reasoning in Problem Solving Process of S3

S3 has written down all the facts contained in the problem. It means S3 understands what given and what has to be found in the problem. This is seen in the answer sheet in Figure 5.

① Diberikan himpunan A dengan operasi perkalian modulo 8. Untuk sebarang $a, b, c \in \mathbb{Z}_8$ jika $ac = bc$ apakah $a = b$? Jelaskan jawaban saudara. Diket $(\mathbb{Z}_8, \times)$ Adib $\forall a, b, c \in \mathbb{Z}_8$ jika $ac = bc$ apakah $a = b$? Jawab $\forall a, b, c \in \mathbb{Z}_8$ $ac = bc$ $acc^{-1} = bcc^{-1}$ $ae = be$ $a = b$	Translate Version Suppose that a set $\mathbb{Z}_8$ under multiplication modulo 8. For arbitrary $a, b, c \in \mathbb{Z}_8$ if $ac = bc$, is $a = b$? Explain your answer! Information: $(\mathbb{Z}_8, \times)$ Question: $\forall a, b, c \in \mathbb{Z}_8$ If $ac = bc$, is $a = b$? Answer: $\forall a, b, c \in \mathbb{Z}_8$ $ac = bc$ $acc^{-1} = bcc^{-1}$ $ae = be \rightarrow a = b$
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Figure 5. Answer of S3

After S3 understood the problem, S3 made the conjecture that is "if $ac = bc$ then $a = b$ is true" by doing the reasoning as in the interview below.

- R : Why did you write the words "inverse" on the answer sheet? Do you feel a statement " $ac = bc$, then $a = b$ " is true?
- S3 : Because the two right and left sides have the same element of $\mathbb{Z}_8$, namely c , Ma'am.

The making conclusion by abductive reasoning of S3 can be described as follows.

(Fact) $\mathbb{Z}_8$ with modulo multiplication 8

$$\forall a, b, c \in \mathbb{Z}_8, ac = bc$$

Inverse multiplication on $\mathbb{Z}_8$

$$\therefore ac = bc \rightarrow a = b$$

S3 performed abductive reasoning by use inverse element concept to the fact, even though there are elements $\mathbb{Z}_8$ that does not have an inverse. Thus, S3 added the wrong concept to the facts. From the set of facts chosen by S3, the conjecture is written in the statement form " $ac = bc \rightarrow a = b$." The making conclusion by abductive reasoning of S3 can be described as follows.

(Fact) $\forall a, b, c \in \mathbb{Z}_8, ac = bc$

$$acc^{-1} = bcc^{-1}$$

$$\therefore a = b$$

The conclusion decided by S3 appears to be valid because it used deductive reasoning, but due to the addition of the fact that there is an inverse of c in $\mathbb{Z}_8$, the conclusion $a = b$ is false.

Abductive Reasoning in Problem Solving Process of S4

S4 has written all the facts contained in the problem by writing " $\mathbb{Z}_8 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}$ ", will be proven if $ac = bc$ whether $a = b$ ". This is seen in the answer sheet in [Figure 6](#).

	Translate Version $(\mathbb{Z}_8, \times)$, for $a, b, c \in \mathbb{Z}_8$ If $ac = bc$, is $a = b$? Answer: $\mathbb{Z}_8 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}$ will be proved if $ac = bc$, is $a = b$? If $a = 2, b = 3, c = 4, a, b, c \in \mathbb{Z}_8$ that $ac = bc$, so that $ac = bc$ $(2.4) \bmod 8 ? (3.4) \bmod 8$ $8 \bmod 8 ? 12 \bmod 8 \leftrightarrow 0 \neq 4$
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Figure 6. Answer of S4

After S4 understood the problem by writing down what is known and asked, then S4 plan to solve the problem by making a conjecture that is if $ac = bc$, then $a = b$ is true as seen from the following interview.

- R : Why did you immediately take $a = 2, b = 3, c = 4, a, b, c \in \mathbb{Z}_8$?
- S4 : So, first, I look for an element of $\mathbb{Z}_8$ then to prove the statement, I take any a, b, c , elements of $\mathbb{Z}_8$, because in the problem it is known "if $ac = bc$ then $a = b$," then I want to prove whether if a, b, c are different, then $ac \neq bc$
- In my answer, I give a "question mark" because I still want to prove whether $ac = bc$ and it was found that $ac \neq bc$.
- R : Which statement do you want to prove?
- S4 : The statement "if $ac = bc$ then $a = b$ "
- R : Oh yes ... So, is that statement true or not?
- S4 : True, Ma'am

So, S4 made the conjecture and statement that "if $ac = bc$ then $a = b$ " is true for the following reasons (as in the interview).

- R : Does that mean you guess that statement is correct? Why?
- S4 : Yes Ma'am
Because of the property of multiplication Ma'am
- R : Could you explain it?
- S4 : If $ac = bc$, then $a = b$. So if I multiply two same numbers to each side, then the result will be the same.
Take any a, b, c member of $\mathbb{Z}_8$ where $a = b$ is consequently $ac = bc$
- $$\begin{aligned} a &= b = 3, c = 2 \\ ac &? bc \\ (3 \times 2) \bmod 8 &? (3 \times 2) \bmod 8 \\ 6 \bmod 8 &? 6 \bmod 8 \\ 6 &= 6 \end{aligned}$$
- So, it is proven that if $ac = bc$, then $a = b$

From the answer of S 4 and interview with S4, the following conclusions can be analyzed

(Fact) $\mathbb{Z}_8 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}$

Properties of multiplication is that if $a = b$ then $ac = bc$

$\therefore$ if $ac = bc$ then $a = b$

The using reasoning in here is not deductive or inductive but abductive, which is following Peirce's abductive theory (Niiniluoto, 2018; Peirce, 1958). After making the conjecture, S4 finds a, b, c so $ac = bc$ results in $a = b$ seen in Figure 7.

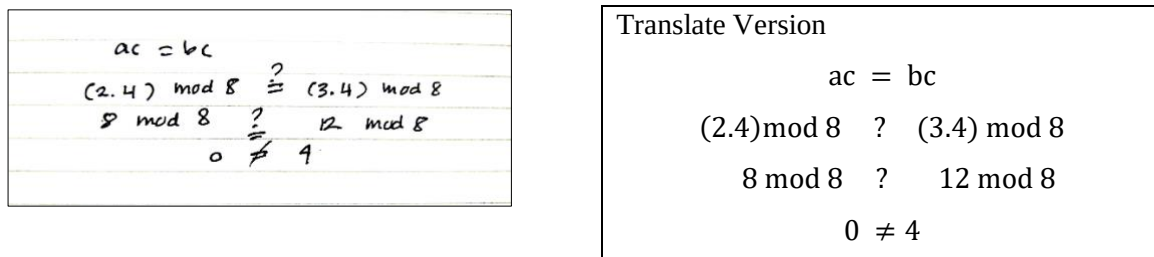


Figure 7. Answer the Problem of S4

From the answer of S4, it shows that the conjecture carried out when understanding the problem makes the problem-solving plan wrong. When teachers already know their mistakes, it will be easy to correct students' mistakes.

The analysis of data in this study documented four types of abductive reasoning based on using the facts of the problem, such as creative conjecture type, the fact optimization type, the factual error type, and the mistaken fact type. Explanation about the characteristics of each type is displayed in Table 1.

Table 1. Characteristics of Students in Abductive Reasoning

Types of Abductive Reasoning based on Using The Fact	Description of Characteristic
First Type: Creative Conjecture	• using all the facts inside the problem to solve the problem • knowing what the questions are asking about, • using facts that "true" outside the problem to solve the problem • Making a conjecture from facts by writing or describing or sketching to design problem-solving • writing a new conjecture outside the question on the problem related to the question
Second Type : Fact Optimization	• using all the facts inside the problem to solve the problem • knowing what the questions are asking about, • using facts that "true" outside the problem to solve the problem
Third Type: Factual Error	• using all the facts inside the problem to solve the problem • knowing what the questions are asking about, • using all the fact to solve the problem • using facts that "false" outside the problem to solve the problem
Fourth Type : Mistaken Fact	• not using the facts inside the problem to solve the problem • not knowing what the questions are asking about, • assuming the questionable thing as a given fact

The uniqueness of the student with "creative conjectures," is writing a new conjecture. The new conjecture is outside the question on the problem related to the question. The fact optimization students solved problems by using all the facts inside the problem, know what the questions are asking about, and using facts that "true" outside the problem to solve the problem knowledge that is not written in the problem (i.e $\mathbb{Z}_8$ is not a group). Because other knowledge taken is the right knowledge and all the facts are known, then the conclusion of the facts is right, so that the planning of the solution is also right.

In the group with "factual error", they used all the facts. When they made a conjecture, this group added other knowledge, that is not contained in the problem, that is the inverse of element concept. The concept that taken is false because not all elements in $\mathbb{Z}_8$ have a multiplicative inverse. Adding error facts to the problem will be the product of a false conjecture so that the planning of the problem is wrong.

In the group with "Mistaken Fact," The students concluded $a = b$ by using the questionable thing as a given fact. The answer is indeed wrong, but it is interesting to analyze because many students do it.

CONCLUSION

This study captures the types of abductive reasoning enacted by students in solving algebra problems related to using facts on the problems. The results showcased some types such as creative conjectures type, fact optimization type, factual error type, and mistaken fact type. In the creative

conjecture type, the students can solve the problem, but they were unsatisfied with the solution. The students develop new ideas related to the questions carried out using abductive reasoning. In fact optimization type, students make conjectures about the answers to problems, then confirm the conjectures by making deductive reasoning. In the factual error types, students add facts outside of the problem to help them solve the problem, but the facts taken are incorrect, leading to the wrong conclusion. In the mistaken fact type, students assume that the question is a true value so that it is appointed as a fact. As a result, the actions taken to make conclusions are also wrong.

The creative conjectures type and the fact optimization type are very useful in the problem-solving process. Therefore, teachers should encourage students to use creative conjectures and fact optimization when learning mathematics. Abductive reasoning can encourage students to find new knowledge. Meanwhile, abductive reasoning can also lead students to make mistakes in the conclusion. By knowing the types of abductive reasoning of the students in solving algebra problems, attentive actions can be taken to correct the errors made and can also provide more activities for student learning so that new knowledge continues to emerge.

This study of abductive reasoning was carried out on mathematics education students, who, at the time of this study enactment, learn both mathematics and pedagogy materials for teaching and learning. The results may be slightly different if it is carried out on students who enroll in the pure mathematics program. Therefore, future investigation is encouraged to examine the abductive reasoning of students in the non-educational programs.

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LEARNING INTEGERS WITH REALISTIC MATHEMATICS EDUCATION APPROACH BASED ON ISLAMIC VALUES

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Abstract

Mathematical learning not only produces students who succeed in mathematical and procedural calculations but also develops religious thinking. Realistic mathematics education with the context of Islamic values makes students can imagine, which is one of the right ways to develop the skills of students' creativity, collaboration, and communication. This study aims to describe the learning trajectory that can help students understand integers with a realistic mathematics education approach based on Islamic values. It is hoped that student responses are positive, meaningful, and enjoyable. This research uses the design research method, which is a form of a qualitative approach. There are three stages in this research, namely: preliminary design, experimental design, and retrospective analysis. The results showed that the Hypothetical Learning Trajectory (HLT) trial with an Islamic value-based context showed significant progress based on student responses. Initially, students had difficulty understanding integers, but they felt delighted to follow the learning process along with the habituation. The HLT technique used in habituation was through pilot experiments, followed by teaching experiments. Students respond very positively and are happy to follow it by seeing the very significant development of their abilities during the learning process.

Keywords: Learning integers, Design research, Realistic Mathematics Education, Islamic values

Abstrak

Pembelajaran matematika tidak hanya menghasilkan siswa yang berhasil dalam perhitungan matematis dan prosedural, tetapi juga mengembangkan pemikiran religius. Pendidikan matematika realistik dengan konteks nilai-nilai keislaman membuat siswa berimajinasi yang merupakan salah satu cara yang tepat untuk mengembangkan keterampilan kreativitas, kolaborasi, dan komunikasi siswa. Penelitian ini bertujuan untuk mendeskripsikan lintasan pembelajaran yang dapat membantu siswa memahami bilangan bulat dengan pendekatan pendidikan matematika realistik berbasis nilai-nilai Islam. Harapannya, respon siswa bersifat positif, bermakna, dan menyenangkan. Penelitian ini menggunakan metode penelitian desain dalam bentuk pendekatan kualitatif. Terdapat tiga tahapan dalam penelitian ini yaitu: desain pendahuluan, desain eksperimental, dan analisis retrospektif. Hasil penelitian menunjukkan bahwa uji coba Hypothetical Learning Trajectory (HLT) menggunakan konteks berbasis nilai Islam memperlihatkan perkembangan yang signifikan berdasarkan hasil respon siswa. Awalnya siswa mengalami kesulitan memahami bilangan bulat, namun mereka merasa senang mengikuti proses pembelajaran beserta pembiasaannya. HLT yang digunakan dalam pembiasaan adalah melalui percobaan percontohan, dilanjutkan dengan mengajar percobaan. Siswa merespon dengan sangat positif dan senang mengikutinya dengan melihat perkembangan kemampuan mereka yang sangat signifikan selama proses pembelajaran.

Kata kunci: Pembelajaran bilangan bulat, *Design research*, Pendidikan Matematika Realistik, Nilai-nilai Islam

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Mathematics has several characteristics, among others. It has abstract studies objects, logical and deductive mindset, and symbols that are empty of meaning. Furthermore, it is systematic and relies on the context. On the other hand, it is based on the agreement/definition and must be studied hierarchically. In connection with the plain meaning of mathematical symbols, the user use needs to concern about the context or scope of the conversation. In the study of mathematics, religion, and

culture, these characteristics provide opportunities that mathematical symbols can be given a specific meaning, especially when the scope of the conversation enters the realm of religion (Abdussakir, 2017).

Mathematics learning includes integers, whose starting point starts from context is Indonesia's realistic Mathematics Education. Integers are an essential requirement for human activities in daily life. Using these numbers, humans can mention many, few, less, more, the same, add, less, times, for, and give value to daily transaction activities. Abdussakir (2014) states that in the Qur'an, there are lessons about 38 different numbers, so there is no doubt that the Qur'an talks about mathematics, mostly about numbers.

That's why it is time for all teachers to start designing every learning, including integers, to enter Islamic values. By doing this, we hope to develop a good character for students. Education has an excellent opportunity to carry out the process of fostering and developing humanity. They do it by delivering teaching methods that are based on noble morals. We call it an education that combines aqidah, shari'ah, and morals in learning. The values are then integrated into the schools' learning process (Nihayati, Suningsih, & Abdullah, 2019). Education in schools must be able to develop students' values through religious education; we can do it by aligning the progress of science and technology (Maafi, 2011). This is then reinforced by the opinion of Abdussakir (2018), who states that Islam does not separate between science and religion. Declining morals and lack of religious values urged the need for reform in education (Rahman, Abdurrahman, & Kadaryanto, 2015; Irawan & Kencanawaty, 2017; Maryati & Priatna, 2017).

However, such ideals have yet to come into realization. Mathematics learning is usually done as a single subject, excluding Islamic values. As a result, mathematics learning becomes rigid, seems complicated, alienated with the reality of life, and ultimately tends to be a subject that students avoid. Mathematics is less able to provide the inculcation of both character and Islamic values. On the other hand, mathematics achievement tends to be low (Salafudin, 2015).

We can see the fact of the low mathematical achievement of Indonesian students in the example as follows. In the International Mathematics Olympiad, which Indonesian students attended every year, only a few students get medals. Likewise, it is found in TIMSS (Third International Mathematics and Science Study) and PISA (International Program for Students Assessment) (OECD, 2013). In TIMSS, our second-grade junior high school students ranked 34 out of 38 countries, while in PISA 2015, we ranked 63 out of 70 countries (Hadi, 2017). Students' ability to solve and interpret problems in various situations is still considered at a low level (Kamaliyah, Zulkardi, & Darmawijoyo, 2013). Such a fact is a significant concern to all parties; therefore, we need to put effort into improving the situation. One action that can be done by the teacher is to enhance learning patterns and the quality of learning. To such an initiative, teachers need to develop effective teaching strategies. Lesson study is a learning system to improve students' pedagogical abilities.

On the other hand, the teachers' pedagogical skill is the most concerning (Susanto, Rozali, & Agustina, 2019; Brandt, Burgener, Barth, & Redman, 2019). This is because pedagogical competency is teachers' ability to manage learning (Rahman et al., 2015; Kunter et al., 2013). This includes teachers'

ability to understand and develop, plan and implement learning, and evaluate learning (Dirgantoro, 2018). But teachers must try to manage the learning system to find the meaning of their education.

Johnson (2002) and Hadi (2017) said that when students find meaning from learning mathematics in school, they will understand and remember what they have learned. Contextual learning enables students to connect learning in school with the real context in everyday life. Contextual knowledge broadens their context. This is because providing new experiences for students can encourage their minds to make new relationships. Consequently, students can discover and construct the meanings on their own.

Constructivism in Mathematics Education approach is one view delivered by the Indonesian Realistic Mathematics Education (PMRI). The right learning strategies and methods are needed to develop students' thinking abilities. It is based on the orientation toward technical skills and mathematics education reform (Irawan & Kencanawaty, 2017; Sembiring, Hadi, & Dolk, 2008). Such reform is based on developing problem-solving skills in daily life. The Indonesian Realistic Mathematics Education recommends such the learning method (PMRI), an adaptation of Realistic Mathematics Education (RME).

One of the contexts used in PMRI was culture (Wahyudi, Zulkardi, & Darmawijoyo, 2016). Kuntowijoyo (2001) stated that religion and culture are two things that interact and influence each other to take shape, symbols, and content/values. Previously, it has been explained that religion and culture provide meaning to certain mathematical symbols. Especially when the context of speech enters the religious realm (Abdussakir, 2017), Islamic value-based context that students often do in their daily lives, this means it can also be used in PMRI with the principle that the real context does not mean concrete physical and tangible, but also includes what can be imagined by students' minds. Real-world problems are essential in RME, and it means students are offered problems they can visualize (Van den Heutvel - Panhuizen & Drijvers, 2014). Problems that students can solve by visualizing it will need students' pedagogical skills.

To improve students' pedagogical abilities, it is necessary to integrate the PMRI approach with lesson study and collaborative learning process (Asari, Fauziah, & Uchtiawati, 2018; Putri, Dolk, & Zulkardi, 2015). Lesson studies are learning systems that use collaborative techniques (Sato, 2014a; 2014b). It is a learning system that was first carried out in Japan's education system (Darra & Kanellopoulou, 2019). It shows promising results in improving the teaching and learning process (Kusumah & Nurhasanah, 2017; Wessels, 2018). The use of lesson study in this learning environment dramatically helps develop a better learning environment. This is because the lesson study ensures that every student gets the same learning (Sato, 2014a). Students work in teams to create collaborative learning tools, and each step in the lessons helps them improve their pedagogical abilities (Coenders & Verhoef, 2019; Fauziah et al., 2019).

Research on the implementation of the use of this realistic mathematics education approach has produced encouraging reports (Ulandari, Amry, & Saragih, 2019). Students become more interested and happier to learn mathematics and show satisfying learning outcomes (Hadi, 2017). From the

description above, the question arises "how to design the learning path that can help students understand integers with a realistic approach based on Islamic values, meaningfully and pleasantly, characterized by positive student responses?"

METHOD

This research uses design research as our research method in this study. The design research consists of five characteristics, such as interventionist, process-oriented, reflective components, cyclic characters, and theory-oriented (Akker et al., 2006; Gravemeijer, 2004; Prahmana, 2017).

Research design is a form of qualitative approach because it emphasizes the socially constructed reality, has a close relationship between researchers and the subjects studied and has an emphasis on process and meaning (Denzin, 2017; Miles, Huberman, & Saldana, 2013). Besides, the design research method is a systematic and flexible method to improve the classroom's quality of learning. It does it by collaborating between researchers and teachers to develop learning designs (Gravemeijer, 1994). The design research method is also an appropriate way to answer research questions and achieve research objectives, namely developing Local Instruction Theory (LIT). It is based on existing theory (theory-driven) and empirically experimented (empirically based) through collaboration between researchers and teachers to increase the relevance of research to educational policies and practices (Gravemeijer & Cobb, 2006). There are four main reasons for using the design research method in this study.

First, there is no theory about the learning trajectories of Islamic value-based realistic approaches to strengthen religious characters. By continuously presenting the context of the Quran-based and Hadith or Islamic values, it supports students to get accustomed to Qur'an and Hadith (Islamic values) in mathematical literacy. In return, this will strengthen the religious character of students with the Islamic values-based Realistic Mathematics Education approach. Second, this study allows researchers to study student learning processes to discover how activities -- with integer learning trajectories with Islamic value-based realistic approaches -- can have an impact on the growth of mathematics literacy based on Islamic values (the Qur'an and Hadith) in understanding integers. In doing so, it strengthens students' religious character, which is consistent with the objectives of this study. Third, to determine how an Islamic value-based context can help students understand integer operations in class VII of Muhammadiyah Palembang Middle School. Fourth, to ensure that students can learn happily and meaningfully.

The development of learning design is carried out in three stages, namely preliminary design, design experiments, and retrospective analysis (Bakker, 2004; Gravemeijer & Cobb, 2006; Simonson, 2006; Prahmana, 2017). The preliminary design aims to design the Hypothetical Learning Trajectory (HLT), which is then refined in the design experiment stage (Prahmana, 2017). Activities carried out in this stage are collaborating with model teachers to conduct a literature review of integer concepts, realistic mathematics education, and Islamic value-based contexts that can be used in integer learning. Also, researchers analyzed the idea of integers in the 2013 mathematics education curriculum.

Furthermore, the results of literature studies and curriculum analysis were used as a basis for designing learning trajectories and developing allegations into HLT.

In this case, the theory serves as a guide to increase students' activities and to think ability in every learning activity, so that it is flexible and can be revised during the experimental design stage. At the design experiment stage, the learning trajectory designed at the initial design stage is then implemented in the learning process (Prahmana, 2017). The purpose of this implementation is to explore and observe students' strategies and thoughts. There are two cycles in this stage; the first cycle is a pilot experiment that aims to evaluate and improve the learning trajectory that has been designed. The second cycle is a teaching experiment that attempts to apply the learning trajectory considered and revised in the first cycle. Implementation of integer learning activities with an Islamic values-based realistic mathematics education approach consists of five activities, shown in the Result and Discussion section.

The last step is a retrospective analysis. All data collected in the design experiment stage were analyzed by comparing the assumptions and HLT with the results of applying the learning trajectory carried out at the design experiment stage (Gravemeijer & Cobb, 2006). From the analysis, we will obtain the work in the form of a description of integers' learning trajectory with a realistic mathematical approach based on Islamic values.

RESULTS AND DISCUSSION

The results of this study describe the learning trajectory of integers with a realistic approach based on Islamic values. Islamic value-based context is a starting point in learning integers and their operations. Activities in implementing the learning path that has undergone improvement are written into five activities. We give a Student Activity Sheet (SAS). SAS is a teaching material format that contains instructions, a list of tasks, and guidance to do activities. There are five SAS according to the number of activities, namely: SAS 1 to SAS 5. We will show the results of the activities of the Islamic value-based context for writing, sorting, and comparing integers, with examples of activities 1 using five SAS, as shown in Figure 1.



Figure 1. Example of activities using SAS

Activities in SAS 1

We distributed SAS 1 to students. Its learning goals are that students can write integers, sort integers, and compare integers. In the beginning, the model teacher opens the lesson in a standard way, which is to draw attention by saying, "This time, you will learn integers by utilizing what is already in your mind and based on Islamic values." The teacher then motivates students by conveying learning objectives that are already listed on the first page of SAS 1 and the benefits after learning this material. Then the teacher gives a reference by reminding the main issues to be discussed. After that, the teacher making meaningful lessons by linking the initial knowledge that is already in the minds of students. The following are examples of the teacher's question to students.

"What can you say about the Holy Qur'an?"

"Can you all read the Qur'an?"

"Do you still lie often?"

"Who can write integers in front of the class?"

In the end, the students start doing activities according to the SAS 1 instructions. The following are the student activities in SAS 1:

1. "This is the Book about which there is no doubt, a guidance for those conscious of Allah "(Surah. Al-Baqarah, 2). Indeed, the Qur'an is guidance and guidance for good. The Qur'an guides those who fear their Lord by obeying His commands, moving away from His prohibitions and leaving disobedience / sinning, then they benefit from it. Those are the three descriptions of the Qur'an. Why do you agree and be sure of the opinions above?

Student answers:

Because everything written in the Qur'an must be true, considering the Qur'an itself was revealed to the prophet Muhammad SWT

Figure 2. Student answers to question number 1 SAS 1

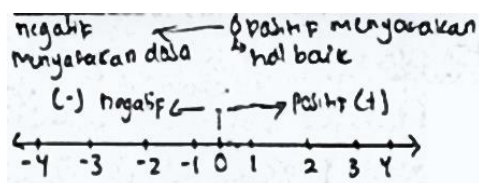
Students' answers to the questions seen in [Figure 2](#), as a sign that students can imagine the context based on the Quran. Students' answers turned out to be different, but all showed their confidence in the truth of the contents of the Qur'an.

2. "And establish prayer at the two ends of the day and at the approach of the night. Indeed, good deeds do away with misdeeds. That is a reminder for those who remember. (Surat Hud, verse 114). If there is no sin, we say it is denoted by 0 (zero), then write an integer by first observing the picture showing night and day seen in [Figure 3](#)!



Figure 3. Earth night and day

Student answers:



negative refers to sin $\leftrightarrow$ positive refers to good deeds

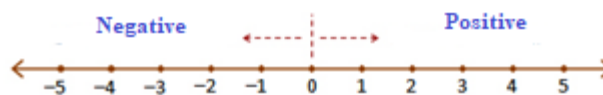


Figure 4. Students' answers to problem number 2 SAS 1

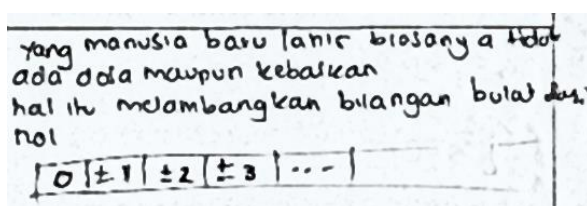
Generally, students can answer like in Figure 4, even though there are still other possibilities of how to write integers without using the help of a number line, as one of the students writing:

-5	-4	-3	-2	-1	0	1	2	3	4	5
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Some others wrote, ..., -2, -1, 0, 1, 2, etc.

3. "No baby is born but upon Fitrah (as a Muslim). It is his parents who make him a Jew or a Christian or a Polytheist." (Sahih Muslim, Book 033, Number 6426). How do you sort integers according to the statement above?

Student answers:



The newly born human usually has no sin or goodness, it symbolizes integers from zero

0	± 1	± 2	± 3	...
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Figure 5. Student answers to question number 3 SAS 1

Figure 5 shows that student's answer is understandable because he has understanding that "For every child born, they are clean from sin, even though after adolescence, they may be doing good or bad". We found out that there were only who wrote: $0, \pm 1, \pm 2, \dots$ while others wrote, $0, -1, +1, -2, +2, \dots$, or $0, -1, 1, -2, 2, \dots$

4. A 15-storey Medina hotel, has 3 underground floors. An employee was on the 3rd floor of the office. He went down 6 floors, then up 4 floors. Which floor number is bigger after going down 6 floors and then going up 4 floors? (use the "<" or ">" sign).

Student answers:

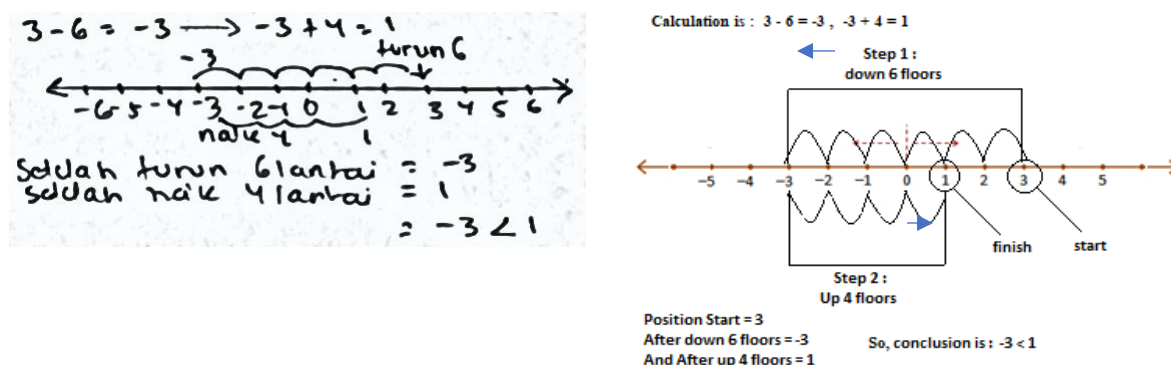


Figure 6. Student answers to question number 4 SAS 1

His answer, shown in Figure 6, was encouraging. It can be seen that he could explore the context of the problem of moving up and down the elevator, illustrated in the number line.

Activities in SAS 2

We distributed Student Activity Sheet 2 (SAS 2) to students. Its learning objective is that students can add integers. In the beginning, the model teacher opens the lesson in a standard way, which is to draw attention by saying, "This time, you will learn integers by utilizing what is already in your mind and based on Islamic values." The teacher then motivates students by conveying learning objectives that are already listed on the first page of SAS 2 and the benefits after learning this material. Then the teacher gives a reference by reminding the main issues to be discussed. After that, the teacher making meaningful lessons by linking the initial knowledge that is already in the minds of students. The following are examples of the teacher's question to students.

"Can you mention Fardlu (mandatory) prayers in the daytime and Fardlu (mandatory) prayers in night?"

"Can you name some good deeds?"

"Who can help answer $5 + (-2)$?"

In the end, the students start doing activities according to the SAS 2 instructions. The following are the student activities in SAS 2:

1. Look again at the meaning of the Q.S. Huud verse 114 this below:

"And establish prayer at the two ends of the day and at the approach of the night. Indeed, good deeds do away with misdeeds. That is a reminder for those who remember. (Q.S. Surat Hud, verse 114).

- a. Can prayers erase bad deeds (sins)? Why?
- b. Which sins can be removed by good deeds?

Student answers:

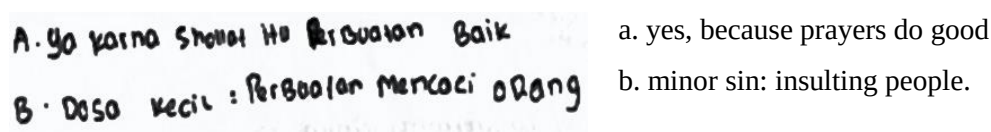


Figure 7. Student answers to question number 1 SAS 2

One other student answered like one in Figure 7. Some answered: a. yes, because prayer is obligatory for every Muslim as a good deed that can wash away small sins, b. Minor sins, lying, insulting friends, stealing, and many others they can name of.

2. Read and comprehend the word of Allah in Surah Al-Kahf, verse 25.

“And they remained in their cave for three hundred years and exceeded by nine”.

- a. Which mathematical operation expressed above?
- b. From question a), why Allah expressed the verse using such mathematical operation? Why didn't Allah show the exact number instead?

Student answers:

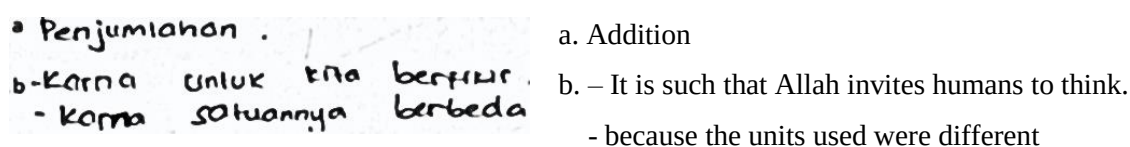


Figure 8. Student answers to question number 2 SAS 2

Using the context shown in Figure 8, we can invite students to discuss about addition, as well as a to think about the meaning of the verse.

3. Express the integer addition operation below by making examples from these ideas.
 - a. Good deeds followed by another good deeds are better than the sum of both deeds
 - b. Good deeds followed by bad deed will decrease the good deed.
 - c. Bad deeds followed by good deeds will decrease the bad deed.
 - d. Bad deeds followed by another bad deeds are worse than the sum of both deeds

Student answers:

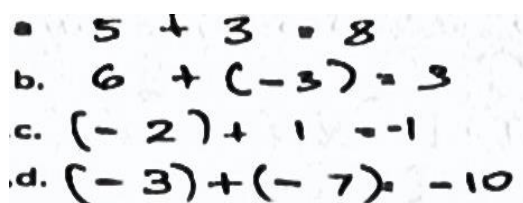


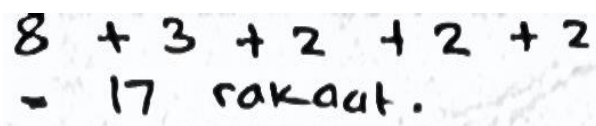
Figure 9. Student answers to question number 3 SAS 2

Question number 3, continuing from the context of Surah Hud:114, was used to guide students to find the concept of addition. It turned out students could answer it as shown in Figure 9.

4. Amir was doing the midnight prayer, ended by witr. He then went to masjid to pray in congregation. He entered the masjid, then doing Tahiyatul Masjid prayer. After that, he stood up to do Qobliyah Subuh prayer. Last, he prayed Subuh.

How many raka'ahs did Amir pray that morning?

Student answers:



$$8 + 3 + 2 + 2 + 2 = 17 \text{ rak'at.}$$

Figure 10. Student answers to question number 4 SAS 2

The teacher should start by explain the Islamic value-based context with open-ended answers, such as in the in Figure 10 above. Even though there are many possible answers because the midnight prayer can be done by the combination of Tahajjud (2,4, a multiplication of 2 rakaah, up to 20 rakaah) and Witr (1, or 3 rakaah). As in Subuh, Tahiyatul Masjid, and Qobliyah Subuh, they have exact number of rakaah.

Activities in SAS 3

We distributed Student Activity Sheet 3 (SAS 3) to students. Its learning objective is that students can subtract integers. In the beginning, the model teacher opens the lesson in a standard way, which is to draw attention by saying, "This time, you will learn integers by utilizing what is already in your mind and based on Islamic values." The teacher then motivates students by conveying learning objectives that are already listed on the first page of SAS 3 and the benefits after learning this material. Then the teacher gives a reference by reminding the main issues to be discussed. After that, the teacher making meaningful lessons by linking the initial knowledge that is already in the minds of students. The following are examples of the teacher's question to students.

"Why did Allah order humans do good?"

"What sins can fardlu(mandatory) prayers erase?"

"Who can help answer 5- (-2)?"

In the end, the students start doing activities according to the SAS 3 instructions. The following are the student activities in SAS 3:

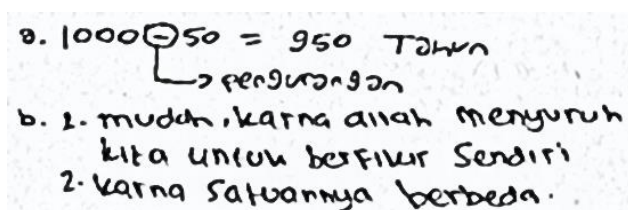
1. Read and pay attention of following:

"And We certainly sent Noah to his people, and he remained among them a thousand years minus fifty years, and the flood seized them while they were wrongdoers" (Surah Al Ankabut:14)

- a. Which mathematical operation expressed above!

- b. From question a), why Allah expressed the verse using such mathematical operation? Why didn't Allah show the exact number instead?

Student answers:



a. $1000 - 50 = 950$

→ subtraction

- b. 1) easy, because Allah tells us to only exercise our thinking ability;
2) because the units are different

Figure 11. Student answers to question number 1 SAS 3

Using the context shown in [Figure 11](#), we can invite students to discuss about addition, as well as a to think about the meaning of the verse.

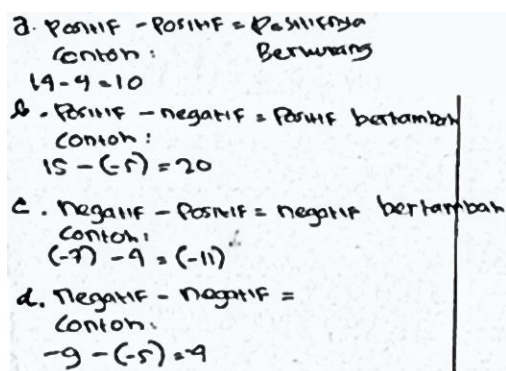
2. Read and pay attention to below hadith!

" Have taqwa (fear) of Allah wherever you may be and follow up a bad deed with a good deed which will wipe it out and behave well towards the people ". (HR. Tirmidhi).

Express the integer addition operation below by making examples from these ideas.

- Someone whose good deeds is taken away from him, it is worse for him.
- Someone who has good deeds and Allah take away his bad deeds, it is better for him.
- Someone who has bad deeds and Allah take away his good deeds, it is worse for him.
- Someone whose bad deeds is taken away from him, it is better for him.

Student answers:



- a. positive - positive = positive decreases

example: $14 - 4 = 10$

- b. positive - negative = positive increases

example: $15 - (-5) = 20$

- c. negative - positive = negative increases

example: $(-7) - 4 = -11$

- d. negative - negative = negative decreases

example: $-9 - (-5) = -4$

Figure 12. Student answers to question number 2 SAS 3

Teachers could use the context shown in the hadith above to guide students to understand subtraction. It turned out students can answer it as shown in [Figure 12](#).

3. The temperature in Mecca is 37°C (above 0°C). At the same time, the temperature in Japan is 40°C below the temperature of Mecca. What is the temperature in Japan now?

Student answers:

$dik = \text{Suhu Makkah} = 37^{\circ}\text{C (di atas } 0^{\circ}\text{C)}$
 $= \text{Suhu Jepang} = 40^{\circ}\text{C (di bawah Makkah)}$
 $= 37^{\circ}\text{C} - 40^{\circ}\text{C} = -3^{\circ}\text{C}$

The temperature in Makkah is 37°C (above 0°C) and

The temperature in Japan at that time is 40°C below the temperature of Makkah.

$$= 37^{\circ}\text{C} - 40^{\circ}\text{C} = -3^{\circ}\text{C}$$

Figure 13. Student answers to question number 3 SAS 3

Problem number 3 was used to evaluate students' understanding of subtraction. In general, students answered as shown in [Figure 13](#).

4. If the Fajr prayer time is at 04.43 and the Dzuhur prayer time is at 12.02, how long do we have to wait from Fajr to Dzuhur time?

Student answers:

$$\begin{array}{r} 12.02 \\ - 04.43 \\ \hline 11.62 \\ - 04.43 \\ \hline 07.19 \end{array}$$

Figure 14. Student answers to question number 4 SAS 3

The student's answer in [Figure 14](#) showed an effort to express time subtraction, from 12.02 to 11.62 (since 1 hour = 60 minutes, then the student subtracts 11.62 with 04.43 = 07.19. It appears that the student only completed the calculation step but yet to answer the question, which is by stating that the time needed to wait from Fajr to Dzuhur prayer is 7 hours 19 minutes.

Activities in SAS 4

We distributed Student Activity Sheet 4 (SAS 4) to students. Its learning objective is that students can subtract integers. In the beginning, the model teacher opens the lesson in a standard way, which is to draw attention by saying, "This time, you will learn integers by utilizing what is already in your mind and based on Islamic values." The teacher then motivates students by conveying learning objectives that are already listed on the first page of SAS 4 and the benefits after learning this material. Then the teacher gives a reference by reminding the main issues to be discussed. After that, the teacher making meaningful lessons by linking the initial knowledge that is already in the minds of students. The following are examples of the teacher's question to students.

"Can you accuse someone? Explain your opinion."

"Do you know, how much Allah reward humans for doing a good deed?"

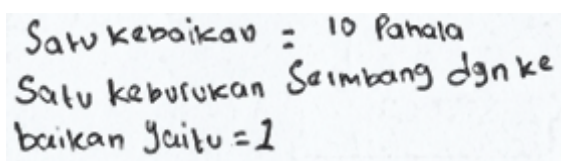
"Why is multiplication called repeated addition? Give an example."

In the end, the students start doing activities according to the SAS 4 instructions. The following are the student activities in SAS 4:

1. Read and pay attention to the following verse:

“Whoever comes [on the Day of Judgement] with a good deed will have ten times the like thereof [to his credit], and whoever comes with an evil deed will not be recompensed except the like thereof; and they will not be wronged” (Surah Al-Anam: 160). From the verse, how much will Allah reward humans for doing good deed? And how much for doing rewards taken away for doing bad deeds?

Student's answers:



one good deed = 10 reward

one bad deed = balanced with the bad he made,
which is 1 evil

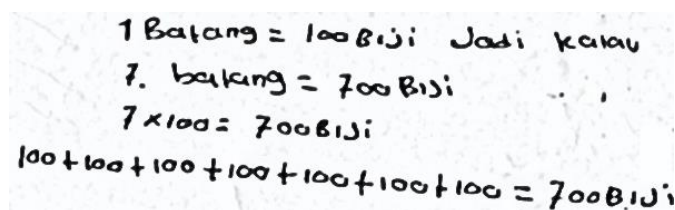
Figure 15. Student answers to question number 1 SAS 4

It turned out that in [Figure 15](#), it is true that each good will be crime with 10 rewards, but one bad will get a penalty of sin in proportion to his crime. It may be that students were in a hurry, such that they wrote goodness instead of bad deed.

2. The example of those who spend their wealth in the way of Allah is like a seed [of grain] which grows seven spikes; in each spike is a hundred grains. And Allah multiplies [His reward] for whom He wills. And Allah is all-Encompassing and Knowing. (Surah Al Baqarah:261)

How do you express the verse above into the mathematical symbol?

Student's answers:



1 tree = 100 seeds.

So, 7 trees = 700 seeds

$7 \times 100 = 700$ seeds

$100 + 100 + 100 + 100 + 100 + 100 + 100 +$

$100 = 700$ seeds

Figure 16. Students' answer to question number 2 SAS 4

[Figure 16](#) shows that the student's answer showed that he had absorbed the meaning of the verse. Although he had yet to express that $100 + 100 + 100 + 100 + 100 + 100 + 100 = 7 \times 100$ (Muslimin et al., 2018).

3. But whoever earns an offense or a sin and then blames it on an innocent [person] has taken upon himself a slander and manifest sin. (Surah Annisa:112). We imply "And whoever earns an offense or a sin" as negative expression and assign negative symbol (-). We imply "blames it on an

innocent [person]" as positive expression and assign positive symbol (+). We imply " ... has taken upon himself a slander and manifest sin " as negative expression and assign negative symbol (-). Note the order of the symbols, "negative, positive, negative". Observe the patterns formed. Express the four different possible patterns using multiplication!

Student answers:

$$\begin{array}{l} \text{a. } - \times + = - \\ \text{b. } + \times + = - \\ \text{c. } - \times - = + \\ \text{d. } + \times - = - \end{array}$$

Figure 17. Student answers to question number 3 SAS 4

Teacher used the context from the verse above to guide students explore the concept of multiplication. It turned out students can answer it as shown in [Figure 17](#).

4. Express the statements below using multiplication. Give examples.

- The act of saying a fact truthfully, is a good deed
- The act of saying a fact falsely, is a bad deed
- The act of saying a lie as if it's a fact, is a bad deed
- The act of saying truthfully, that something is a lie is a good deed

Student's answers:

$$\begin{array}{l} \text{a. } 5 \times 2 = 10 \\ \text{b. } -10 \times 5 = -50 \\ \text{c. } -5 \times 5 = -25 \\ \text{d. } -10 \times 10 = 100 \end{array}$$

Figure 18. Student's answer to question number 4 SAS 4

Students' answers to question number 4 shown in [Figure 18](#) above, are analogous from [Figure 17](#).

Activities in SAS 5

We distributed Student Activity Sheet (SAS 5) to students. Its learning objective is that students can multiply integers. In the beginning, the model teacher opens the lesson in a standard way, which is to draw attention by saying, "This time, you will learn integers by utilizing what is already in your mind and based on Islamic values." The teacher then motivates students by conveying learning objectives listed on the first page of SAS 5 and the benefits after learning this material. Then the teacher gives a reference by reminding the main issues to be discussed. After that, the teacher makes meaningful

lessons by linking the initial knowledge that is already in students' minds. The following are examples of the teacher's question to students.

"Do you still remember that multiplication is called repeated addition? Give an example!"

"Mom makes a small circle shaped cake. If she divides the cake equally to her three children, how many pieces does each child get?"

"How much is $\frac{1}{3}$ of Rp 15,000.00?"

In the end, the students start doing activities according to the SAS 5 instructions. The following are the student activities in SAS 5:

1. Read the following verse

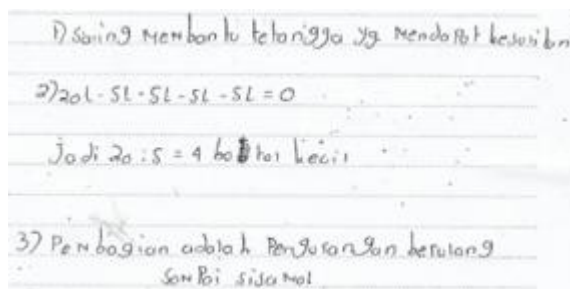
"Whoever believes in Allah and the Last Day, he should glorify his neighbors" (HR. Bukhari 5589, Muslim 70)" or

" O Abu Dharr, when you cook a stew, put more water in the broth and take care your neighbors." (HR. Muslim).

Pay attention to the words of the Prophet above which are narrated by Bukhari and Muslim, then:

- Write your opinion from reading the hadith above. Then, think the following context.
- Mom buys 20 liters of cooking oil. She wants to share with her neighbors, so she pours the oil into small 5-liter bottles. How many small bottles she needs?
- So, what do you think division is?

Student' answers:



- 1) Often helps neighbors who get into trouble
- 2) 20 liters - 5 liters - 5 liters - 5 liters - 5 liters
= 0
- 3) The division is a recurring subtraction to remainders zero;

Figure 19. Student Answers Number 1a in SAS 5

Students' answers in Figure 19 showed that a hadith-based context let students to imagine in their minds. One of them answered, "helping neighbors who are in trouble", even though many other answers available. When students visualized the action of pouring cooking oil into small bottles, then he, indirectly, performed a repeated subtraction. this is used to strengthen the understanding that the positive integer division operation is a repeated subtraction operation. Finally, students can calculate as shown in Figure 19.

2. $4: -\frac{25}{10} \leftrightarrow 4 \times (-2,5) = \dots?$
3. $6: 2 = 3 \leftrightarrow \frac{1}{2} \times 6 = 3$. Calculate $7500: 5 = \dots? \leftrightarrow \frac{1}{5} \times 7500 = \dots?$

Student's answers:

Figure 20. Student answers to question number 1b SAS 5

As shown in [Figure 20](#), students understood that division is an inverse multiplication operation. To support students' understanding, multiplication can also be interpreted as a repeated addition (Hendriana, Prahmana, & Hidayat, 2019).

4. Suppose a husband dies, leaving a wife, a mother and a son. His inheritance is 48 million rupiah. It is known that the share for a wife is $1/8$, mother gets $1/6$ share, son got the remaining share. How much do each of them get?

Student's answers:

a. Ahli Waris	b. Bagian	d. Siham	e. Harta warisan yang diperoleh
Istri	$1/8$	3	$3/24 \times \text{Rp } 48 \text{ juta} = 6.000.000$
Ibu	$1/6$	4	$4/24 \text{ juta} = 8.000.000$
anak laki laki	Sisa = $17/24$	17	$17/24 \times 48 \text{ juta} = 34.000.000$
c. Majmu' Siham = Asal masalah/KPK		24	Jumlah harta warisan Rp 48 juta

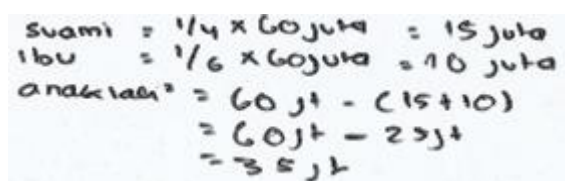
Heir	Share	Siham (divider) = 24	Amount of assets obtained
Wife	$1/8$	3	$3/24 \times \text{Rp } 48 \text{ million} = 6,000,000$
Mother	$1/6$	4	$4/24 \times \text{Rp } 48 \text{ million} = 8,000,000$
Son	Remaining	17	$17/24 \times \text{Rp } 48 \text{ million} = 34,000,000$
		24	= Rp 48 million

Figure 21. Student answers to question number 2 SAS 5

[Figure 21](#) shows that student understood that division is an inverse multiplication operation and vice versa. In this exercise, student uses two number operation namely division and multiplication to solve this problem.

5. Suppose a wife dies, leaving a husband, a mother, and a son. Her inheritance is 60 million rupiah. It is known that the share of each heir; husband $1/4$, mother $1/6$ and remaining for her son. How much does each of them receive?

Student's answers:



Handwritten student work showing calculations for family members' ages:

$$\begin{aligned} \text{suami} &= 1/4 \times 60 \text{ juta} = 15 \text{ juta} \\ \text{ibu} &= 1/6 \times 60 \text{ juta} = 10 \text{ juta} \\ \text{anak laki-laki} &= 60 \text{ jt} - (15 + 10) \\ &= 60 \text{ jt} - 25 \text{ jt} \\ &= 35 \text{ jt} \end{aligned}$$

$$\text{Husband} = 1/4 \times 60 \text{ million} = 15 \text{ million}$$

$$\text{Mother} = 1/6 \times 60 \text{ million} = 10 \text{ million}$$

$$\text{son} = 60 - (15 + 10)$$

$$= 60 - 25 = 35 \text{ million}$$

Figure 22. Student's answer to question number 3 SAS 5

Figure 22 explains that student understood that division is an inverse multiplication operation and vice versa. Furthermore, student also understands that addition is an inverse subtraction operation and vice versa. Lastly, based on this exercise, student can use four number operation namely addition, subtraction, division, and multiplication to solve this problem.

By closely evaluating the implementation of Learning Trajectory (LT) above, results shown from SAS 1 through SAS 5 activities are by the five characteristics of PMRI. This is the adoption of Realistic Mathematics Education (RME) (Treffers, 1987; Wijaya, 2012; Daryanto, 2012; Maslihah, 2012). From the results and the previous discussion, several findings were obtained as follows.

Firstly, every time a meeting begins with the activity of reading the Qur'an or Hadith, students who can and especially those who are fluent in reading the Qur'an have tried to get them appointed. While those who have not and cannot read the Qur'an fluently, they choose to be set to read the meaning. Secondly, at the initial meeting, students have not been active in doing activities. They may not be accustomed to active thinking themselves or are not accustomed to learning given a context problem, let alone the context based on the Qur'an and Hadith.

Third, from the activity results in the second step, in this third step, they continue the group discussion. It turned out that the discussion activity showed quite well, even though at the beginning of the conversation, the students were still confused and awkward. Yet, after a few minutes of discussion with the teacher's direction, they had begun to be active. The results of this study add to the empirical evidence which states that Islamic values can be integrated with the mathematics learning process which can emerge student understanding (Lubis, 2015; Masduki, Khotimah, & Sutarni, 2014; Muhsinin, 2013; Muslimin et al., 2019; Piliang, Daulay, & Siddik, 2017).

Lastly, at the end of each meeting, there is a class discussion. That is, they present the results of each group's discussion. At the first meeting, students are asking "Why at night there are many crimes (sinning)?", All members of the group from each group fell silent. Then the teacher told them to open the Q.S. Al-Falaq, please open the Qur'an translation, they open the Qur'an translation, and some are reading from their handphone. Suddenly someone half shouted "I know sir!", Without being appointed, he immediately called "Q.S. Al-Falaq verse 3 sir, meaning" and from the crime of the night when it was pitch black". It shows that starting to enjoy the activity of discussion towards the action in the activities,

as mentioned earlier, is inseparable from the words do good and do evil (sin). If that is done by the teacher, especially the mathematics teacher, the students will learn really and enthusiastically. Sometimes, they will indirectly realize that learning requires compulsory education (Asy-Syaimaa' Hussain & Ramli, 2017). There is interference from their God, so they do not dare to be arrogant, let alone not be polite to the teacher.

CONCLUSION

The learning trajectory of integers through context-based on Islamic values was effective in helping students understand integers. The proven learning trajectories consist of four phases, namely beginning the presentation of the context-based on Islamic values, *Iqra* (literate) to understand the context, resolving the context individually continued in groups to develop an informal model to formal, and communicating with the presentation.

In addition, the learning trajectory of integers with a starting point context based on Islamic values, in addition to sharpening students' minds, focus, meaningful, and fun as well as developing good character. If the teacher wishes to do mathematics learning in general with a learning trajectory using Islamic value-based contexts, then he must accept the challenge of learning to understand the Qur'an and Hadith.

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A COMPARISON OF MATHEMATICAL TASKS TYPES USED IN INDONESIAN AND AUSTRALIAN TEXTBOOKS BASED ON GEOMETRY CONTENTS

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Abstract

This study examined the type of mathematical tasks in two Australian and two Indonesian mathematics textbooks for 7th-grade students. The quantitative data were collected from the coding results of the tasks in the textbooks. The tasks were coded based on six categories: the presentation forms, the cognitive requirements, the contextual features, the information provided, the number of steps required, and the numbers of answers. Both the similarities and differences in the mathematical tasks provided in the selected textbooks were analysed. The coding results reveal that the majority of tasks in both the Australian and Indonesian textbooks were presented in verbal and combined forms. Routine and closed tasks were still dominant in the four textbooks. More than 93% of tasks in the four textbooks had sufficient information for students to solve the problem. One of the Australian textbooks had a higher proportion of tasks with real-world contexts than the other textbooks. One of the Indonesian textbooks showed a high proportion of tasks requiring multiple steps or procedures. These results were used to explore the learning opportunities offered by the textbooks, and the possible influence on students' performances in international assessments. Some recommendation for the refinement of the textbooks and future research are also outlined at the end of the study.

Keywords: Textbook analysis, Mathematical task, Australia, Indonesia

Abstrak

Penelitian ini meneliti jenis soal matematika pada dua buku teks Australia dan dua buku teks Indonesia untuk siswa kelas 7 (tujuh). Data kuantitatif diperoleh dari hasil pengkodean pada keempat buku teks tersebut. Soal-soal dikoding berdasarkan pada enam kategori: bentuk tampilan soal, syarat pengetahuan yang diperlukan, fitur kontekstual, informasi yang tersedia, banyak tahapan (langkah) penyelesaian, dan banyaknya jawaban yang mungkin dari soal tersebut. Persamaan dan perbedaan dari keempat buku teks tersebut dianalisis. Hasil pengkodean menunjukkan bahwa kebanyakan soal pada buku teks Australia dan Indonesia disajikan dalam bentuk verbal dan bentuk gabungan. Soal rutin dan soal tertutup masih mendominasi keempat buku teks. Lebih dari 93% soal pada keempat buku teks memberikan informasi yang cukup bagi siswa untuk menyelesaikan soal tersebut. Salah satu buku teks Australia memiliki lebih banyak soal dengan konteks dunia nyata dibandingkan dengan tiga buku teks lainnya. Salah satu buku teks Indonesia memiliki proporsi yang tinggi untuk tipe soal yang harus diselesaikan dengan banyak langkah (tahapan). Hasil ini digunakan untuk mengeksplorasi peluang belajar yang ditawarkan oleh buku teks, dan kemungkinan pengaruhnya pada kinerja siswa dalam penilaian internasional. Beberapa rekomendasi untuk penyempurnaan buku teks dan penelitian di masa depan juga diuraikan di akhir penelitian.

Kata kunci: Analisis buku teks, Soal matematika, Australia, Indonesia

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The performance of Indonesian students in international assessments has become one of the main concerns of the Indonesian Government and mathematics researchers in Indonesia. This can be seen from several studies that have been conducted related to this case (Ahyan, Zulkardi, & Darmawijoyo, 2014; Edo, Hartono, & Putri, 2013; Novita, Zulkardi, & Hartono, 2012). The result of TIMSS 2011 and

2015 show that Indonesian students were still below the ‘Low International Benchmark’ (Mullis et al., 2012; Mullis et al., 2016). The results of PISA 2009, PISA 2012, and PISA 2018 also revealed more than 75 % of Indonesian students were only able to work up to level 2 (OECD, 2010; 2014; 2019). According to research that has been conducted to study this issue, one of the main factors that might influence this performance is the lack of students’ skills for high level or non-routine problems since teachers only used low-level tasks in the mathematics classroom (Ahyan et al., 2014; Edo et al., 2013; Novita et al., 2012). This idea is also reflected by the statement of the Indonesian Ministry of Education and Culture (2013) that a lot of tasks or problems presented in the international assessments were not included in the Indonesian mathematics curriculum.

Some studies have shown that the different results among the countries which participated in the international assessment are related to the differences of quality and quantity of mathematics instructions or textbooks (Mayer, Tajika, & Stanley, 1991). The important role of textbooks in mathematics learning has been recognized and has received attention from the international mathematics education community especially since the inclusion of the analysis of hundreds of textbooks from almost 50 countries in the Third International Mathematics and Science Study in 1995 (Yan & Lianghuo, 2006). One of the ways to investigate and evaluate the textbooks used in one country is by comparing the textbooks with textbooks from other countries. The comparative analysis of textbooks has become a debatable issue among researchers. Some researchers have argued that the analysis of textbooks gives insignificant influence in determining what students learn, since textbooks are part of an intended curriculum, not an implemented curriculum (Freeman & Porter, 1989). However, other researchers believe that the study of textbooks used in different countries can give insight and reveal the similarities and differences of opportunities in mathematics learning provided to students around the world (Charalambous et al., 2010).

In this study, Indonesian textbooks were compared to Australian textbooks, which focused on the types of mathematical tasks provided in the textbooks. Based on the scores in TIMSS 2011, Australian students achieved higher score in all domains compared to Indonesian students (Mullis et al., 2012). Meanwhile, in terms of hours per year, Indonesian students spent more time than Australian students in learning mathematics. Indonesian students spent 173 hours per years, while Australian students only spent 143 hours (Mullis et al., 2012). This fact leads to a question “with fewer hours, what make Australian students achieved higher scores than Indonesian students in international assessment?”. One of the ways to answer this question is by looking at learning instructions provided in these two countries so that we can see the kind of learning opportunities offered for the students.

The aim of this study is to extend the previous research related to Indonesian students’ performance in international assessments, focusing on the comparative analysis of mathematical tasks types in mathematics textbooks from Indonesia and Australia. This study is expected to provide insights into the similarities and differences of the types of mathematical tasks in Indonesian and Australian textbooks. Thus, it will be useful for mathematics teachers, so that they can be more selective in

choosing tasks to provide better learning opportunities to students. Furthermore, it is hoped that this study could provide feedback and suggestions to the Indonesian Government and textbooks developers for the improvement and refinement of mathematics textbooks in Indonesia.

METHOD

Textbooks Selection

The data sources of this study are two Indonesian and two Australian (Victorian) mathematics textbooks for 7th grade presented in Table 1. We chose the first Indonesian textbook because this textbook was published by the Indonesian Ministry of Education and has been used as the main source for many schools that have implemented the new curriculum in Indonesia. This textbook is available online and can freely accessed and downloaded by the teachers and the students. Furthermore, this textbook has been revised several times to meet the international standards.

Table 1. Selected Textbooks

Country	Selected Textbooks
Indonesia	Textbook 1 As'ari, A. R., Tohir, M., Valentino, E., Imron, Z., & Taufiq, I. (2016). <i>Matematika SMP/MTs Kelas VII semester 2</i> (Revised ed.). Jakarta: Pusat Kurikulum dan Perbukuan, Balitbang, Kemdikbud
	Textbook 2 Adinawan, M. C., & Sugijono. (2013). <i>Matematika untuk SMP/MTs kelas VII semester 2</i> . Jakarta: Erlangga
Australia	Textbook 1 Smith, C., Elms, L., Roland, L., & Rowland, R. (2014). <i>Maths Quest 7 for Victoria</i> (Australian Curriculum ed.). Queensland: John Wiley & Sons Australia, Ltd
	Textbook 2 Brown, P., Evans, M., Gaudry, G., Hunt, D., McIntosh, J., Pender, B., & Ramagge, J. (2011). <i>ICE-EM mathematics: Year 7 book 1</i> (Australian curriculum ed.). Melbourne, VIC: Cambridge University Press

The second mathematics textbook-released by a private publisher- was selected based on discussions with several mathematics teachers in Indonesia. In Indonesia, the selection of mathematics textbooks often occurs at the school level hence each school has different mathematics textbooks. Therefore, one of the best ways to determine the most popular textbook is by asking the mathematics teachers.

Two Australian textbooks are the best-selling mathematics textbooks from “the largest supplier of education resources to primary and secondary schools in Australia”, *Campion*. The two textbooks were chosen based on the discussion with the supplier and one mathematics teacher in Victoria. In this research, the two Australian textbooks, Math Quest and ICE-EM, will be mentioned as Aus_1 and

Aus_2, and the two Indonesian textbooks, National textbook and mathematics textbook by Erlangga, will be mentioned as Indo_1 and Ind_2.

Procedures for Textbook Analyses

The chosen topics for this study were geometry since it is one of the content domains assessed in international assessment. Furthermore, Tatsuka et al. (2004) also found that there is a high correlation between geometry and the attribute to measure higher order mathematical thinking skills.

Geometry topics covered in this study are measuring angles, perimeter and area of a square, rectangle, parallelogram, and triangles. After deciding the topics, the researcher marked all of the tasks in each textbook that would be coded in the next step. In Aus_1, the researcher marked all of the tasks under the section “exercise”, “challenge”, “rich task”, and “code puzzle”. In Aus_2, all of the tasks were under the section “exercise”, “review exercise”, and “challenge exercise”. In Ind_1, I marked all of the task under the section “Ayo kita pahami” (Let’s understand), “Ayo kita coba” (Let’s try), and “Ayo kita latihan” (Let’s practice), while in Ind_2, all of the tasks were under the section “Latihan” (Practice), “and Tes Kompetensi” (Competency Test).

Data Analysis

Research related to the type of mathematical tasks has been conducted by many researchers. Li (2000) categorized the tasks based on the number of procedures required for the solution, contextual features, response type, and cognitive requirement. Meanwhile, Guven et al. (2016) distinguished the tasks based on 1) presentation (verbal or visual, containing many data or not, long or short); 2) content (routine or non-routine, irrelevant data or not, incomplete data or not, close to or far from daily life, dependent or independent from the curriculum); and 3) solution (many operations or not, different strategies or not, hard or easy)

In the present research study, the types of mathematical tasks were adopted from a previous study conducted by Yan and Lianghuo (2006) since these types of tasks are more suitable for textbook analysis which cover three main components i.e. presentation, content and solution, explained in Table 2.

Table 2. Types of Mathematical Tasks

Presentation	Content	Solution
Purely Mathematical Forms (M)	<i>Cognitive Requirements</i>	<i>Steps Required</i>
Verbal Form (Vb)	Routine (R)	Single Step (SS)
Visual Form (Vs)	Non-Routine (NR)	Two Steps (TS)
Combined Form (Cb)		Multiple Steps (MS)
	<i>Contextual Features</i>	<i>Numbers of Answers</i>
	Real Context (RC)	Open-Ended (OE)
	Non-Real Context (NRC)	Closed (C)
	<i>Information Provided</i>	
	Sufficient Data (S)	
	Extraneous Data (E)	
	Insufficient Data (I)	

1. 'Tasks in a Purely Mathematical Form, Tasks in a Verbal Form, Tasks in a Visual Form, and Tasks in a Combined Form'

If the presentation of the tasks only included mathematical expression, then they were categorized as tasks in a purely mathematical form. If the presentation of the tasks was in written words only, then they were classified as tasks in a verbal form. If the presentation of the tasks consisted of pictures, graphs, figures, etc., then they were coded as tasks in a visual form. If the presentation of the tasks was a combination of two or three of the above, then they were categorized as tasks in a combined form.

2. 'Routine Tasks vs Non-Routine Tasks'

A routine task is a task that can be solved by merely following a certain known algorithm, formula or procedure, while a non-routine problem is the contrary of routine problem; it cannot be solved by simply using a certain known algorithm, formula, or procedure.

3. 'Real Context Tasks vs Non-Real Context Tasks'

A real context task is a task related to everyday life or real-world context. In contrast, the non-real context is defined as a task that is unrelated to any practical backgrounds in daily life or real-life situation. A task, such as games or puzzles, also belongs to real-context tasks since it can be found in students' everyday life.

4. 'Sufficient Data Tasks, Extraneous Data Tasks, and Insufficient Data Tasks'.

According to Yan and Linghuo (2006), sufficient data tasks mean that the information provided is sufficient for solving the tasks. Extraneous data tasks mean that the information provided is more than enough to solve the task. On the other hand, insufficient data tasks mean the information provided is not enough to obtain the solution of the task.

5. 'Single Steps Tasks, Two Steps Tasks, and Multiple Steps Tasks.'

A Single step task is defined as a task that can be solved by using one direct operation, while a two steps task is a task that needs two steps of procedure to find the solution. Furthermore, if the task needs more than two direct steps, then it belongs to multiple steps tasks.

6. 'Open-ended Tasks vs Closed Tasks'

There are two definitions of the open-ended task: 1) a task that has many possible solutions (Kwon, Park, & Park, 2006); and 2) a task that can be solved using a variety of procedures (Sanchez, 2013). This study focuses on open-ended tasks as tasks that have several or many correct answers (the first definition). Therefore, the closed task is a problem that only has one solution.

An example of the coding used is presented below

Jane is a landscape gardener who is laying a new lawn. The rectangular lawn is 13 m long and 8 wide. How many square meters of turf should Jane order? What is the total cost of the turf if it costs \$12.50 per square meter?

Coding: Verbal Form (V), Routine (R), Real-Context (RC), Sufficient Data (S), Two Steps (TS), Closed-ended (CO)

In order to make the counting process easier, Microsoft excel was used in the coding process. Related to the reliability of the coding process, the researcher used an intra-coder process in which the coding process was carried out twice. The interval time of the first coding and second coding process was approximately one month. The results of these two coding processes showed that there was a slight difference in the first and the second coding especially in the category of routine vs non-routine tasks and the steps required for solving the tasks. The figures that used for the final results were from the second coding because the accuracy in the second coding was better than in the first one. The differences in the results of these two coding processes are less than 5% which means that the coding result of this study is reliable.

RESULTS AND DISCUSSION

General Finding

In this study two Indonesian and two Australian mathematics textbooks for 7th-grade students were analyzed. In general, the four textbooks had different characteristics. The two Australian textbooks, Aus_1 and Aus_2, were bigger, thicker and had more pages than the Indonesian textbooks. The total numbers of tasks in the topics of angles, perimeter and area provided in the textbooks can be seen in [Figure 1](#). It shows that the Australian textbooks had more tasks than the Indonesian textbooks. However, this study focused more on the different types of tasks than the total number of tasks provided in the textbooks.

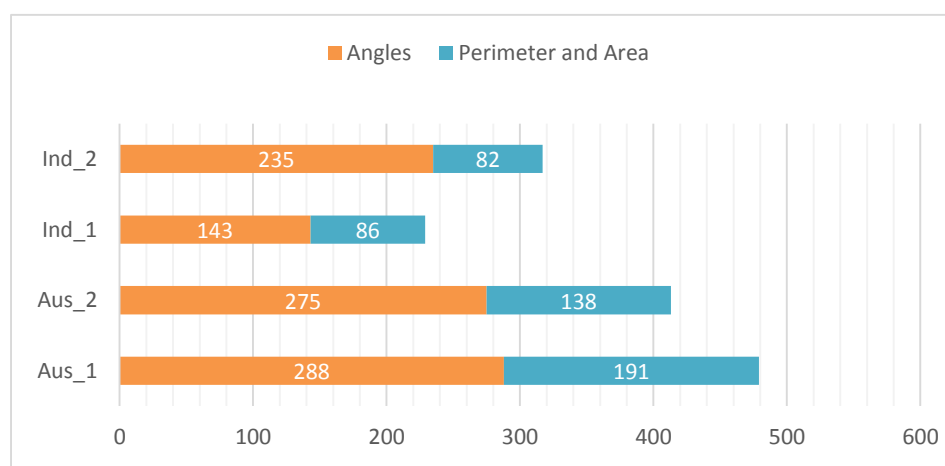


Figure 1. The total number of tasks in the textbooks

Presentation Forms of The Types of Tasks in the Textbooks

The presentation of tasks in the four textbooks was coded into four different presentation forms. The coding result can be seen in [Figure 2](#).

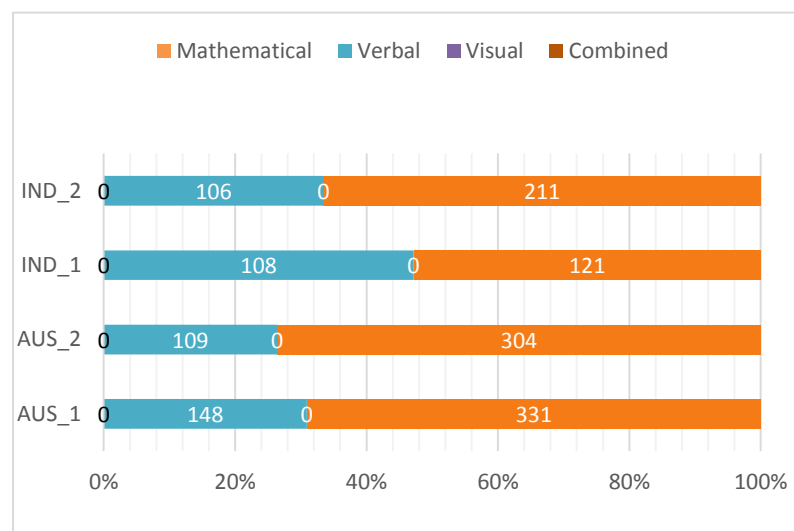


Figure 2. The presentation forms of tasks in the textbooks

The tasks presented in these four textbooks had common types of representation: verbal and combined forms. More than half of the tasks analyzed in this study were presented in the combination of two or three forms, such as the combination of visual and verbal, verbal and purely mathematical form, or verbal, visual and mathematical form. From Figure 2., it can be seen that the Australian textbooks had more combined tasks compared to the Indonesian textbooks. In total, there were 331 out of 479 or 69.1% in Aus_1 and 304 out of 413 or 73.6% in Aus_2 that contained combined forms. Indonesian textbooks were also dominated by the combination representation with 52.8% and 66.6% percentage in the textbook Ind_1 and Ind_2 respectively. The absence of tasks in purely mathematical and visual forms is due to the tasks in these topics always including a verbal form. The most common combined form in these four textbooks is a combination of verbal and visual (i.e., tables or figures), and Aus_1 had more colorful figures compared to other textbooks.

Routine vs Non-Routine

Related to the content, the tasks were coded into two types: routine or non-routine tasks. The distributions of routine and non-routine tasks in the four textbooks can be seen in Figure 3.

The coding results reveal that the majority of tasks in both the Australian and Indonesian textbooks comprised routine tasks. The two Australian textbooks had similar portions of routine tasks: 93.5% in Aus_1 and 95.6% in Aus_2. This result aligns with the study done by Vincent and Stacey (2008) which revealed that four Australian textbooks had a high proportion of problems with low procedural complexity, ranging from 73% to 96% for the topic of geometry. The two Indonesian textbooks, Ind_1 and Ind_2, showed a big difference in the total number of routine tasks.

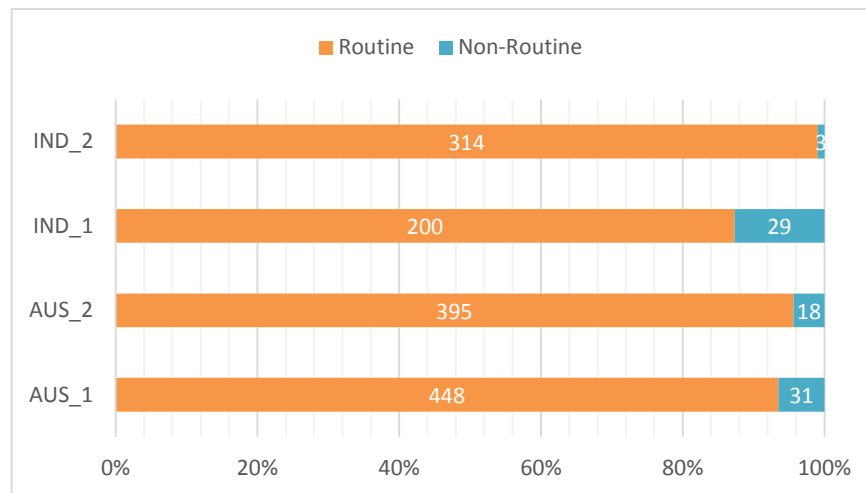


Figure 3. The presentation forms of tasks in the textbooks

Ind_1 contained 87.3% routine tasks, while Ind_2 consisted of 99.1% routine tasks, the highest percentage of the four textbooks, as shown in Figure 3. The difference in publication years of these two textbooks could be one of the reasons that explaining the big gap in routine tasks in the Indonesian textbooks. Ind_2 was published in 2013, while Ind_1 was revised and released in 2016 with the aim of providing instructional materials for teachers that meet international standards.

What length of party lights is needed to decorate the perimeter of a rectangular tent with dimensions 15.5 m by 8.75 m?

Figure 4. Routine task from Aus_1

Some of the routine and non-routine tasks below are taken from the Australian and Indonesian textbooks. The tasks in Figure 4 and Figure 5 were coded as routine tasks since students can solve them simply by using the formula of perimeter and area of rectangle and square.

The area of a rectangle is equal with the area of a square that has the side length 4 cm. Find the length of the rectangle!

Figure 5. Routine task from Ind_2

The tasks in Figure 6 and Figure 7 were coded as non-routine tasks because students cannot use the concept of angles directly. At first, the tasks seem to lack of information, but when the students manipulate the figure by dividing it into several triangles, they will find the solution.

The types of non-routine tasks in the Indonesian textbooks seemed more challenging compared to the Australian textbooks, even some of the non-routine tasks in Ind_1 was taken from the National Mathematics Olympiad questions. Furthermore, several questions in Ind_1 can't be solved by using only one mathematics concept.

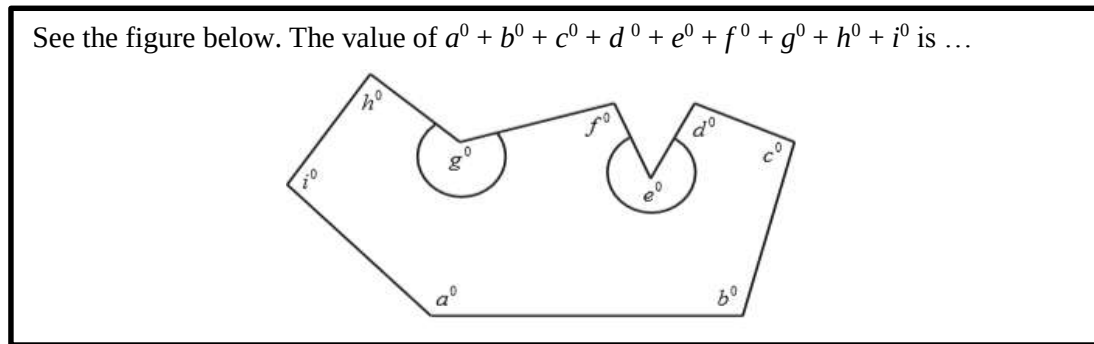


Figure 6. Non-routine task (Adopted from Ind_1)

The task in Figure 8, for example, besides using the concept of the area of the triangle, the students need to connect it with the concept of congruent triangles which will be taught in the 9th grade. Ind_1 was released by the Indonesian Ministry of Education, and the intention to provide tasks (such as that shown in Figure 8) in this textbook must be appreciated since they want Indonesian students to be more familiar with this kind of task.

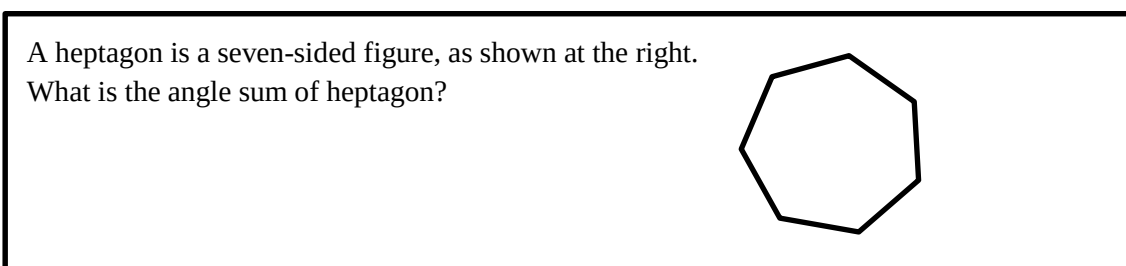


Figure 7. Non-routine task (Adopted from Aus_2)

However, McInerney (2014) proposed that there should be a level of challenge in the task that is suited to students' capacities which is known as the optimal challenge level. Such tasks are neither too easy that makes them look purposeless nor too difficult that makes them look impossible for the students.

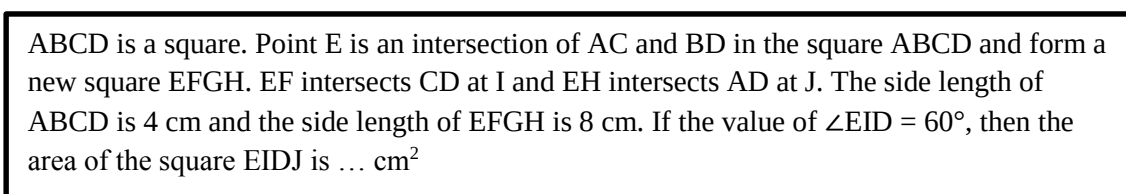


Figure 8. Non-routine task from Ind_1

Real Context vs Non-Real Context

The distribution of real-world context and non-real-world context tasks in the four textbooks are shown in Figure 9. It shows that two Australian textbooks, Aus_1 and Aus_2, had a big difference in terms of the proportion of tasks with real-world contexts. Aus_1 has 26.9% of real-world context tasks, while in Aus_2, more than 95% of tasks were set in non-real contexts and only 4.6% tasks presented

real world situations. The two Indonesian textbooks had less than 20% of tasks that were contextualized in real-world situations (17% and 11% for Ind_1 and Ind_2 respectively). This result is consistent with the findings of a study by Wijaya et al. (2015) which revealed that only 11% of tasks in Indonesian mathematics textbooks were context-based.

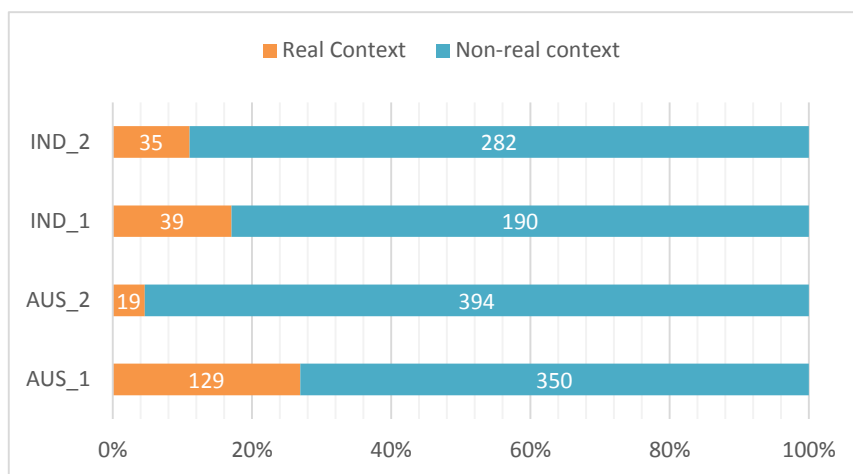


Figure 9. real-context and non-real-context tasks in the textbooks

The high proportion of tasks with real-world contexts in one of the Australian textbooks was predictable since this country has included mathematical literacy, known as ‘numeracy’ in its curriculum. The Australian Education Council (as cited in Goos, Stillman, & Vale, 2007) stated that there are four reasons why a mathematics curriculum is needed in schools: 1) mathematics is useful in daily life; 2) mathematics is necessity for being an intelligent member in civic life; 3) mathematics is useful in a wide range of careers, and 4) mathematics is part of cultural heritage. These reasons reveal that to support students to realise that mathematics is important for their lives, teachers should relate it to everyday situations such as shopping, cooking, and others daily activities (Brady & Bowd, 2006). Similarly, Sullivan (2011, p.3) stated that to set the goals of mathematics teaching; we have to consider “what mathematics is” and “what might be the purposes of teaching mathematics to student”. The results of this study reveal that although Australia has included mathematical literacy in the curriculum, there is still an Australian mathematics textbook, Aus_2, which only provides a small number of tasks with real-world contexts. The tasks below are the samples of tasks with the real-world context in Australian (Figure 10) and Indonesian (Figure 11) textbooks.

The design of tasks related to the real-world context in Aus_1, Aus_2, and in Indonesian textbooks was slightly different. Compared to the other textbooks, the tasks in Aus_1 were more colorful, and most of the tasks used the real figures of the context. On the contrary, most of the tasks with real-world contexts in Aus_2 and in the Indonesian textbooks were presented in verbal type without figures, or using an illustration to resemble the real figure. Even in Aus_2, there was only one task out of 19 tasks with a real-world context that used an illustration.

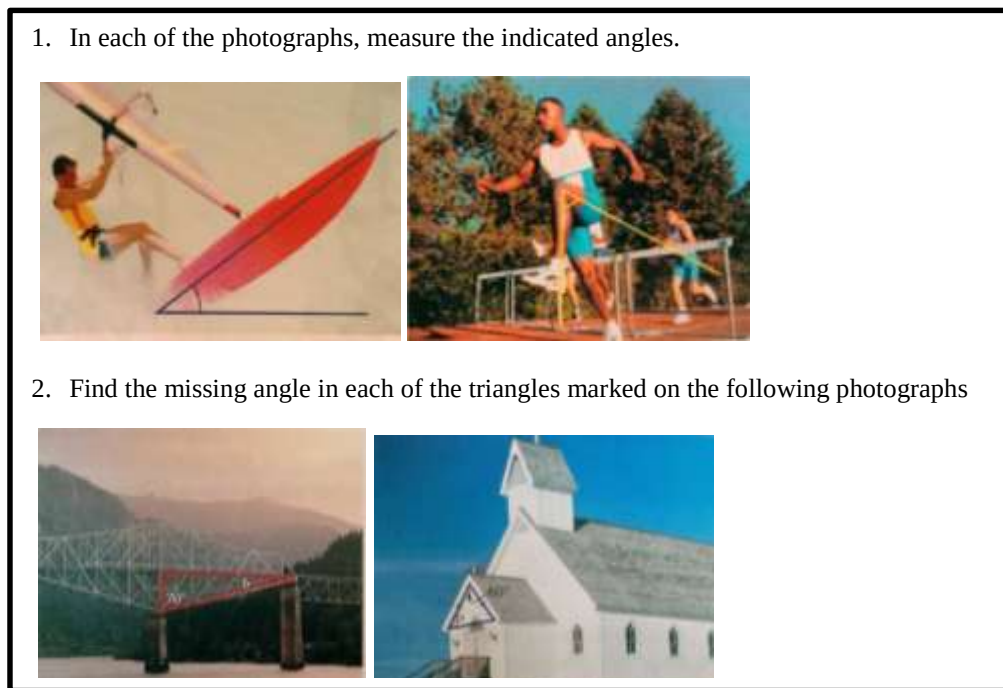


Figure 10. Real context task from Aus_1

Source: Smith, Elms, Roland, & Rowland, 2014, p. 151, 171

Furthermore, the contexts of tasks in Aus_1 were more varied than the contexts in the Indonesian textbooks. In learning the topic of angles, for example, Aus_1 used many contexts such as a clock, playing golf, doing ballet, surfing, flying a kite, windmills, popular buildings in Australia, a Ferris wheel, bicycle and other contexts. Aus_1 also provided tasks that were presented in the games mode which could make students feel more engaged since they might feel that they are playing a game when solving the task. Furthermore, the contexts used in Ind_2 were limited. Some of them were presented repetitively (e.g., clock and stairs). In Ind_1, there were more varied contexts compared to Ind_2, such as archery, a laptop computer, a chair, playing billiards, rails, and the points of a compass. However, these contexts were still less varied compared to Aus_1.

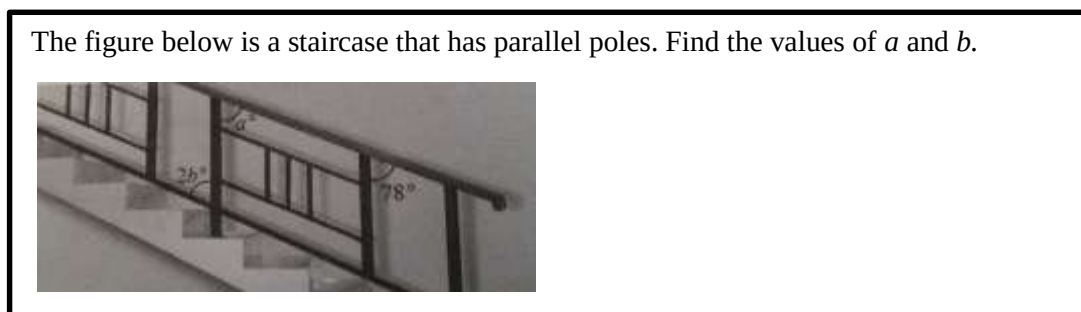


Figure 11. Real context task from Ind_2

Source: Adinawan & Sugijono, 2013, p. 21 – 22

Sufficient Data, Extraneous Data, and Insufficient Data

Related to the information provided in the tasks, the tasks in the four textbooks were differentiated into three types; those are tasks with sufficient data, tasks with extraneous data and tasks with insufficient data. Figure 12 presents the distribution of tasks in the Australian and Indonesian textbooks based on different types of information provided.

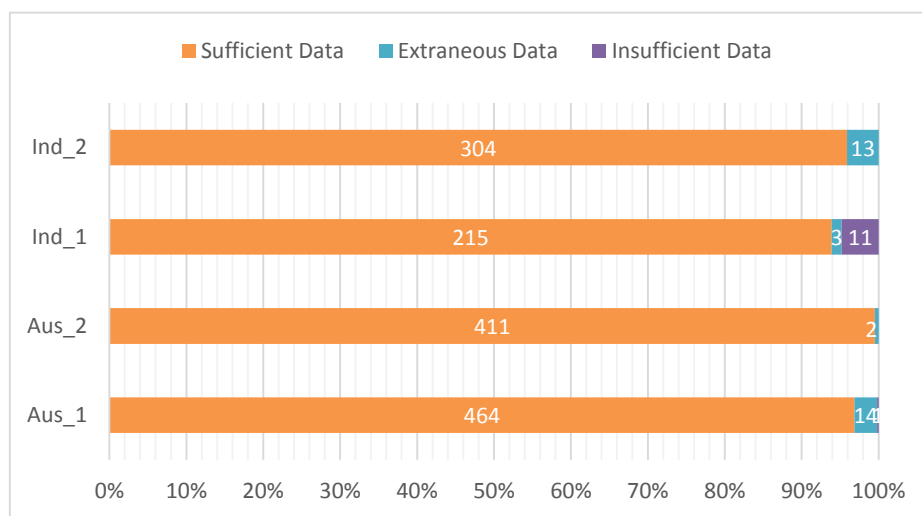


Figure 12. Types of information provided in the textbooks

As shown in Figure 12., more than 93% of tasks in the four textbooks had sufficient information for students to solve the problem. Aus_2 had the highest percentage, that is, 99.5%, followed by Aus_1, Ind_2, and Ind_1 with 96.9%, 95.9%, and 93.9% respectively. All of the textbooks also had a small number of tasks with superfluous data, and the kinds of the extra information in all of the textbooks was quite similar (see Figure 13). From the tasks in Figure 13, it can be seen that there is some unnecessary information for solving the tasks, for example, the length of the hypotenuse. However, these superfluous data are needed to see students' understanding of the concept of the height and base of triangles. The tasks with extraneous information are also effective to reveal students' misconception of certain mathematical concept. Yan and Lianghuo (2006) stated that many students tend to use all the numbers provided in the tasks without knowing whether the numbers are needed to find the solutions or not. Hence this type of task is necessary to help the students learn how to select appropriate information correctly.

In addition, even if the extraneous data in the task will not be used in the process of finding the solution, the information should be realistic and appropriate to the mathematics concept. The tasks numbered (1) and (3) in Figure 13 are examples. The length sides of the triangles are mathematically not realistic and they conflict with the concept of Pythagoras Theorem. Hence, the developers of the textbooks should be more careful when presenting such superfluous information, except there is another intention for providing this conflicted data, for instance, to provoke students' critical thinking.

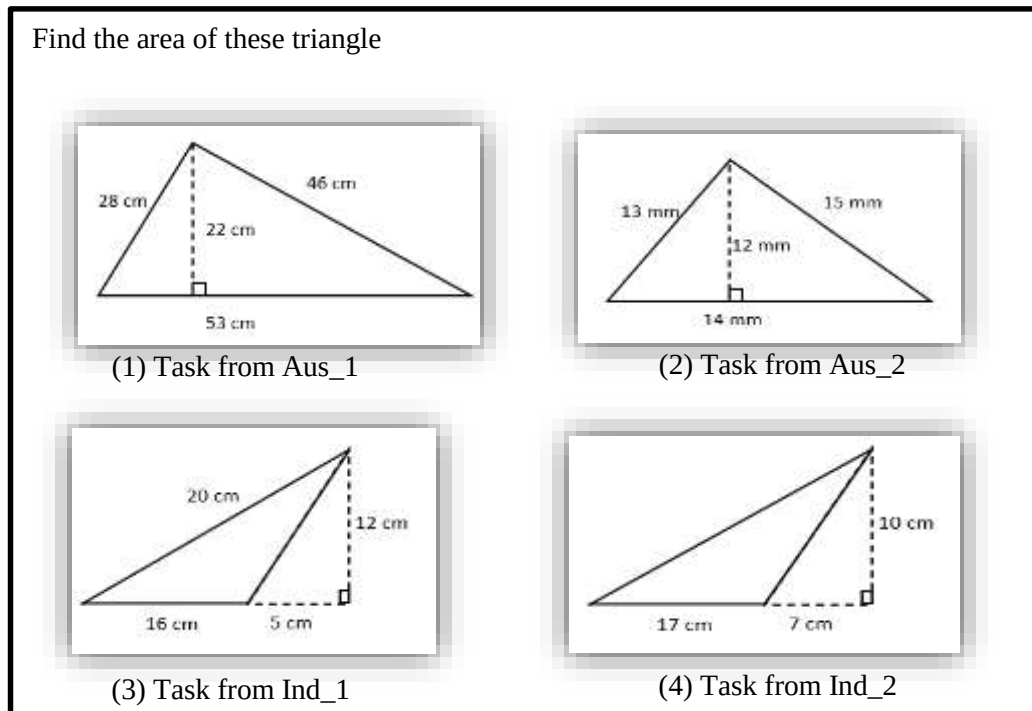


Figure 13. Tasks with superfluous data
Adopted from Aus_1, Aus_2, Ind_1, and Ind_2

Tasks with insufficient information were only found in Aus_1 and Ind_1. In Aus_1, there was one task that did not provide enough information, while in Ind_1, there were 11 out of 229 tasks that contained insufficient data. One task in Aus_1 and four tasks lacking information in Ind_1 cannot be solved (see Figure 14) and 7 tasks in Indo_1 that do not provide enough data can only be solved by making assumptions (see Figure 15).

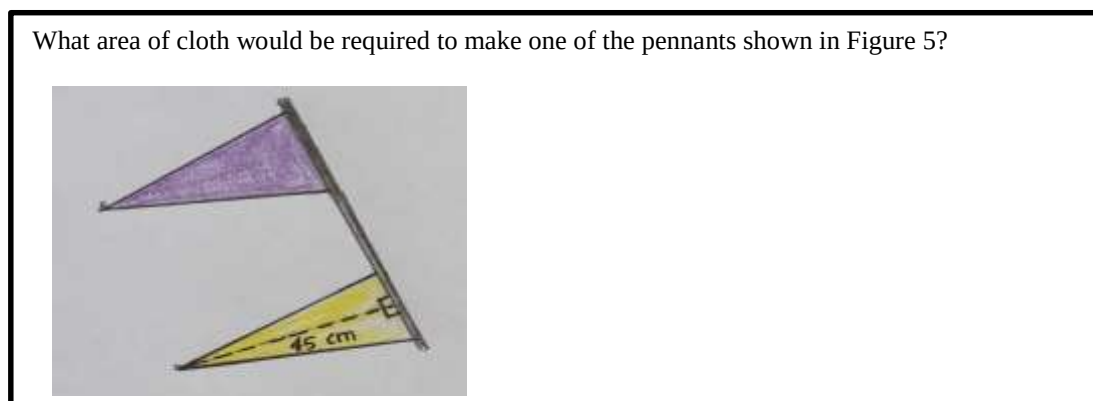


Figure 14. Tasks with insufficient data
Adopted from Aus_1

The task in Figure 14 cannot be solved since there was no information about the length of the base. Meanwhile, the task in Figure 15. was coded as insufficient data because of the lack of parallel signs in the lines provided. The lack of information related to the sign of parallel lines can cause positive and negative effects. For students that have a deep understanding about the concept of parallel lines,

this task can be used to provoke their critical thinking. On the other hand, students who just follow the procedures or formula are likely not to notice it and think that all of the lines that look parallel are parallel without thinking about the importance of the parallel signs.

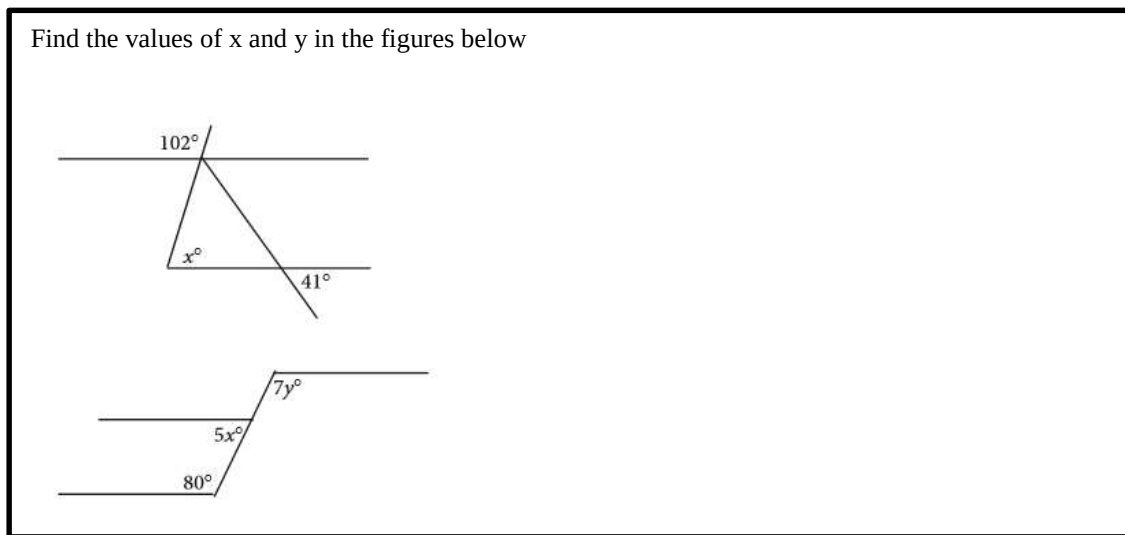


Figure 15. Tasks with insufficient data
Adopted from Ind_1

Single Step, Two Steps, and Multiple Steps

The first category related to solution is the number of steps required to solve the task. In this study the number of steps were classified into three types: single step, two steps, and multiple steps. The coding results of the tasks in the four textbooks based on the number of steps required in finding the solutions can be seen in [Figure 16](#).

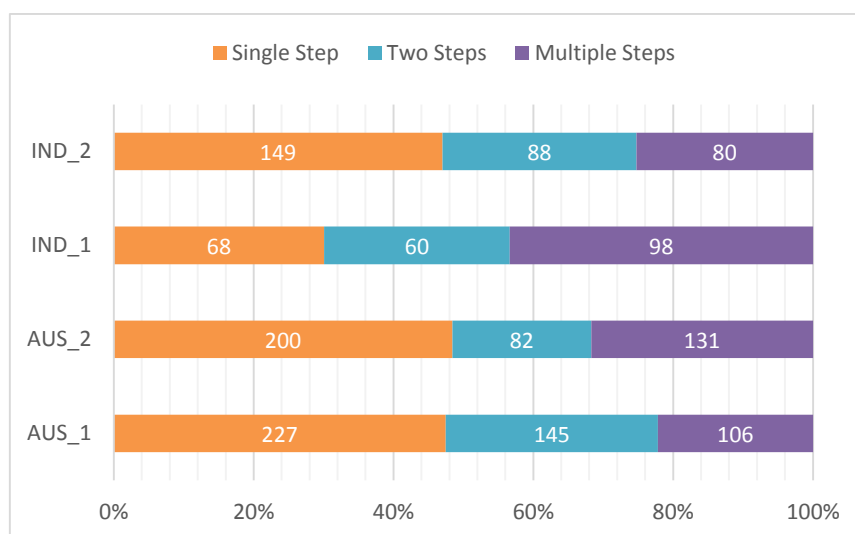


Figure 16. The number of steps required in finding the solution

In terms of the number of steps required to solve the tasks, this study revealed that one of the Indonesian textbooks, Ind_1, seemed more challenging than the other three textbooks, Aus_1, Aus_2, and Ind_2. More than 42% of the tasks in Ind_1 required many steps to solve, while such tasks were

approximately 22% in Aus_1, 31% in Aus_2, and 25% in Ind_2. The two Australian textbooks revealed a high percentage of tasks, over than 40%, requiring a single step procedure. This finding aligns with the results of research by Gatabi et al. (2012) in their comparative study of Australian and Iranian textbooks. They found that there were few tasks that required multiple steps in Australian textbooks

The somewhat lower proportion of multi-step problems in the Australian textbooks might also be seen as further evidence that the Australian textbooks provide more material for less able students. (Gatabi et al., 2012, p. 416)

The Australian textbooks, especially Aus_2, showed a high proportion of tasks requiring a similar step or procedure repetitively (see Figure 17). This finding is consistent with the results of research by Gatabi et al. (2012) which revealed a high degree of repetition in two Australian textbooks with under 10% of tasks requiring a new procedure. Vincent and Stacey (2008) in their study of mathematics textbooks in Australia, also found that 71% of geometry tasks in the textbooks were repetitive. According to Gatabi et al. (2012), this repetition is aimed at students who have a wide range of abilities, especially those who need much more practice than other students. An example is shown in Figure 17.

1. a. Draw a large capital Z and mark two alternate angles with α and β
- b. Draw a large capital N and mark two alternate angles with α and β
- c. Draw a large capital H and mark two co-interior angles with α and β
- d. Draw a large capital H and mark two alternate angles with α and β
- e. Draw a large capital F and mark two corresponding angles with α and β

Figure 17. Task with repetition in Aus_2

The types of multiple step tasks in Australian and Indonesian textbook were also quite different. In the Indonesian textbooks, many tasks required students to find the solutions without giving procedures or clues, while in the Australian textbooks, some of the tasks consisted of sub-questions that guide the students to find the answers, and some of the tasks provide students with a hint (see Figure 18). The red line in the task, shown in Figure 18, is a hint to solve the problem (the insertion of another parallel line).

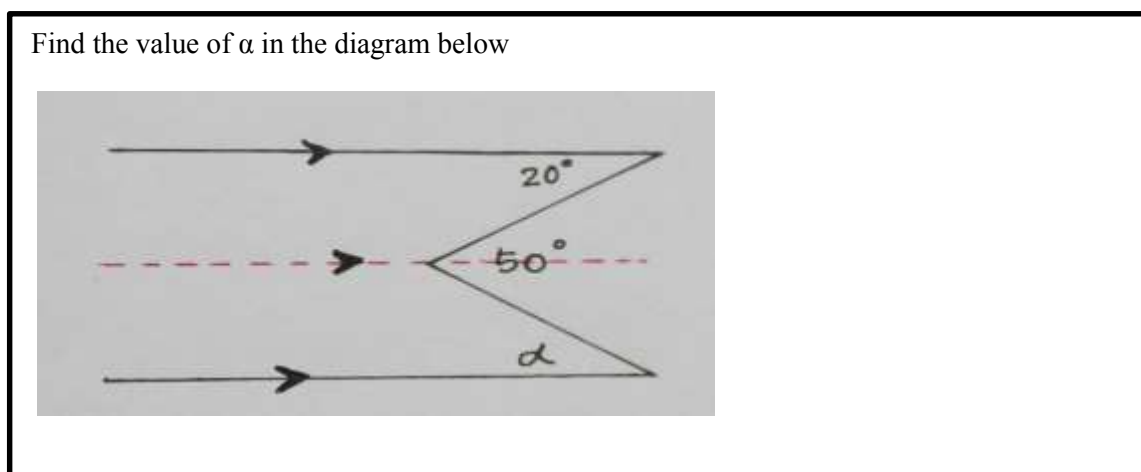


Figure 18. Task with a given clue (red line)
Adopted from Aus_2

Open-ended vs Closed

The last category is based on the number of answers, divided into two types: open-ended task and closed task. A closed task was defined as a task that has one solution and an open-ended task was a task that has more than one solution. The distribution of open-ended and closed tasks in the four textbooks is shown in Figure 19.

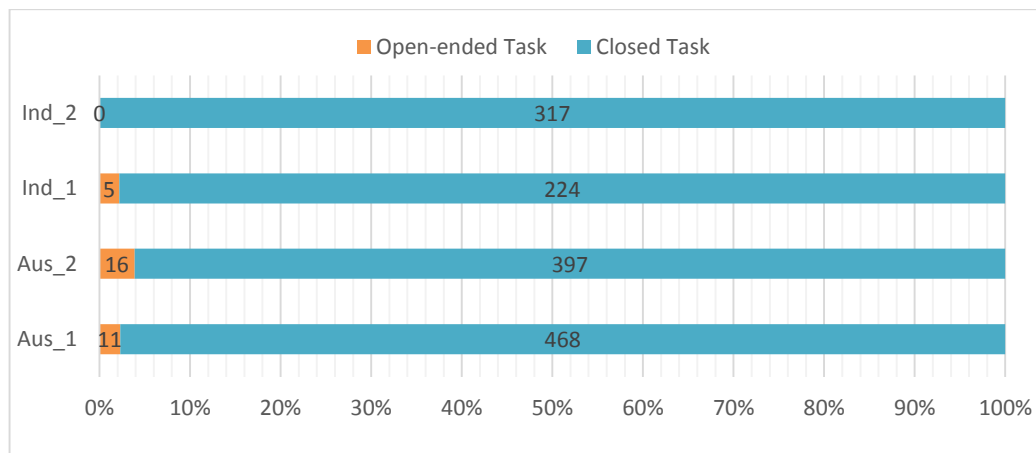


Figure 19. The number of answer (open-ended vs closed)

The majority of tasks in the four textbooks were close-ended questions. One of the Indonesian textbooks, Ind_2, had 100% close-ended tasks. These results reveal that open-ended questions are still not common in both countries. Overall, the Australian textbooks had more open-ended tasks, 3.9% in Aus_2 and 2.3% in Aus_1 than the Indonesian textbooks. One Indonesian textbook, Ind_1, only had 2.2% open-ended questions. The tasks below in Figure 20 and Figure 21 are examples of open-ended questions in the Australian and Indonesian textbooks.

Find one example in your classroom or home of each of the six types of triangles described in this chapter. Describe clearly where the triangle occurs, draw the triangle and classify it according to both side and angle types.

Figure 20. Open-ended task from Aus_1

In Aus_1, some tasks required students to provide their own reasoning. The provision of tasks with this kind of mode of reasoning in Australian textbooks might be one of the factors leading Australian students to achieve higher scores in the reasoning domain in TIMSS 2011 than Indonesian students.

The low number of open-ended tasks in both countries should be a major concern since open-ended tasks give many benefits to students. Boaler (2013) suggested that in order for mathematics to be considered a 'learning subject' rather than just a 'performance subject', teachers should provide open-ended tasks. Cline (2005) also supported this idea by proposing that open-ended tasks are an effective tool to challenge the smartest students without leaving the weaker students behind. In addition, when students try to solve tasks in their own ways, there are likely to be many different ways and many different solutions. This type of situation can encourage students to explain the reasons for their choices through discussions with their friends. Beside

enhancing students' participation in classroom, open-ended tasks has the potential to also cultivate students' mathematical communication skills (Kwon et al., 2006).

1. Could it be a square has the same area as a rectangle? If yes, find the side length of the square and the rectangle.
2. A square is divided into four parts which have the same size and shape. The perimeter of each parts is 16 cm. Find the area of the first square before it was divided.
(Draw about 3 or 4 ways to divide the square and calculate the area of each of the square)

Figure 21. Open-ended task from Ind_1

This study examined the types of mathematical tasks in two Indonesian and two Australian mathematics textbooks for seventh graders. The tasks were coded based on the presentation, the content and the solutions. In general, this study revealed that the Australian textbooks had more tasks than the Indonesian textbooks. Furthermore, the results showed that the majority of tasks in both the Indonesian and Australian textbooks for geometry topics were presented in combined and verbal forms. The combined form was more dominant in the Australian textbooks. In addition, more than 85% of tasks in the four textbooks were routine tasks, that is, could be solved simply by using certain known formulas or procedures. This study also revealed that there was a big difference in the total number of real-world context tasks in the two Australian textbooks with 26.9% and 4.6% in Aus_1 and Aus_2 respectively. It shows that even though numeracy has been included in the Australian curriculum, there is still a textbook that only provides a small number of real-world context tasks. On the other hand, the Indonesian textbooks were still dominated by non-real context tasks, 83% in Ind_1 and 89% in Ind_2.

In terms of information provided in the tasks, the majority of the tasks in the four textbooks provided adequate information. A small number of tasks with extraneous data were found in the four textbooks. However, some of the extra data were not mathematically accurate, for example, the side length of a right-angled triangle. Furthermore, from the results of the coding, it was seen that more than 47% of tasks in the two Australian textbooks could be solved in a single step operation. The study also showed that both Australian and Indonesian textbooks were dominated by more than 95% closed tasks. This means that the proportion of open-ended tasks in the textbooks was less than 5%; one Indonesian textbook, Ind_2, did not include any open-ended tasks.

Many researchers have agreed that practical mathematics, -commonly known as numeracy in Australia, - should be the major focus of the school mathematics curriculum in the compulsory years (Sullivan, 2011). Martin (2007) also asserted that when students raise the question of “when are we ever going to use this?” it means that they do not see the relevance and the importance of what they are learning. Thus, understanding the reason why they need to gain mathematical skills and see the relevance of the knowledge in other subjects and in the world outside the classroom has the potential to enhance their motivation to learn mathematics (OECD, 2014).

Both the Indonesian and Australian textbooks could also include more tasks with missing or extraneous information, because this type of tasks can encourage students to consider the contexts used

and make them learn to choose the relevant information that is needed to solve the problems (Wijaya et.al., 2015). Furthermore, the provision of open-ended tasks may provide many benefits to students and teachers since it is considered as an effective tool to challenge the smartest students without leaving the weaker students behind (Cline, 2005). The open-ended tasks are also dedicated to completing the deficiencies of closed tasks; one of them is disclosing students' misconceptions (Cline, 2005). This means that through the use of the open-ended tasks, teachers will get much more information about students' ways of thinking about a concept in mathematics (Sanchez, 2013; Pehkonen, 1997).

Lastly, as explained by Stein et al. (1996), the nature of mathematical tasks in the textbooks can be changed as they are implemented in the classroom because the implementation is influenced by many factors such as teachers' goals, teachers' knowledge, classroom condition and teachers' instructional habits and dispositions. Therefore, studies focusing on how teachers select tasks from the textbooks, and how the tasks are implemented in the classroom, are needed to complement textbook analyses.

CONCLUSION

Both the Australian and Indonesian textbooks have provided various types of mathematical tasks. However, there are several aspects that could be improved to create better learning opportunities for students. Compared to one of the Australian textbooks, the Indonesian textbooks contained fewer tasks contextualized in the real-world context, whereas this type of task always used in the international assessment such as PISA and TIMSS. Thus, it would be better if the Indonesian Government and textbook developers in Indonesia could include more tasks representing real-world situations and enhance the variation of such real-world contexts. Furthermore, it is also recommended that Indonesian and Australian textbooks should include more non-routine tasks that is suited to students' capacities and open-ended tasks. Lastly, even though the comparative analysis of textbooks can be one of the ways to reveal the types of mathematical tasks experienced by students across countries, it would be better if there is further research on the implementation of the textbooks in the classroom.

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HOW STUDENTS WORK WITH PISA-LIKE MATHEMATICAL TASKS USING COVID-19 CONTEXT

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Abstract

School students can use a sequence of contextual tasks to learn mathematics. We can use Covid-19 as a phenomenon or context to exploit in learning mathematics. This article describes how students learn with mathematical problems that adapted PISA tasks and used the Covid-19 context. This study involved 29 secondary-level students, 15 years old, and each has different levels of mathematical skills. We use three phases of design research as the research method. Data were collected using observation, interviews, and documents. Then, they were analyzed descriptively. The result showed there were ten problems developed, and students were asked to work with those problems. We found that there are steps in how students understand and solve the problem. First, if students find a picture in the task, then they observe at the picture, read the question, and then start working to solve the problem. Second, if students find a table with less data, students refer to all data in solving the problem. Third if students find a table which has a lot of data, then some students calculate all of the data and other only compared among them. We'd like to encourage students to understand the problem before solving the problem. They do this by observing the pictures, comprehending the tables and also the questions.

Keywords: Students' work, Mathematical task, PISA task, Covid-19 context, Design research

Abstrak

Soal matematika yang menggunakan konteks dapat digunakan untuk merangsang siswa dalam belajar matematika. Covid-19 sebagai suatu fenomena merupakan salah satu konteks yang dapat dimanfaatkan. Artikel ini mendeskripsikan bagaimana siswa belajar dengan soal yang telah diadaptasi dari soal PISA dengan menggunakan konteks Covid-19. Penelitian ini melibatkan 9 siswa sekolah menengah, usia 15 tahun, dan kemampuan matematika mereka berbeda. Penelitian desain dipilih sebagai metode penelitian. Pengumpulan data dilakukan melalui observasi, wawancara, dan dokumen serta dianalisis secara deskriptif. Hasil penelitian menunjukkan ada 10 soal yang dirancang dan siswa diminta untuk mengerjakan soal tersebut. Jika soal melibatkan gambar maka siswa melihat gambar tersebut, melihat soal, dan mulai menyelesaikan soal; Jika soal memuat tabel dengan data yang tidak terlalu banyak, maka siswa mengacu pada semua data dalam menyelesaikan masalah; Jika tabel berisi banyak data, maka sebagian siswa menghitung semua data dan yang lainnya hanya membandingkannya. Sebelum siswa mulai memecahkan masalah, mereka mencoba memahami apa yang dimaksud dengan masalah tersebut dengan membaca atau melihat gambar, tabel, dan pertanyaan.

Kata kunci: Pekerjaan siswa, Soal matematika, Soal PISA, Konteks Covid-19, Penelitian desain

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Higher order thinking, is the skill needed by students to face challenges in the future (Ahonen & Kinnunen, 2015; Hesse, Care, Buder, Sassenberg, & Griffin, 2015). These skills include the ability to analyze, evaluate, and be creative (Brookhart, 2010; Stacey et al., 2015). Meanwhile, to cope with the 21st century, students need to master the ability to think critically, be creative, collaborate, and communicate (Ahonen & Kinnunen, 2015; Hesse et al., 2015). Learning mathematics in schools has an important role in constructing these skills. Through mathematics learning, students have opportunities to construct abilities such as problem solving, reasoning, and communicating mathematically (Ahonen

& Kinnunen, 2015).

The fact is, that the skills of Indonesian students are still dominant in lower order thinking skills (Stacey et al., 2015; OECD, 2015; 2019). We can see this from examining the PISA results of Indonesian students (OECD, 2015; 2019). Students still have difficulty in solving math problems that require higher order thinking skills. This is because Indonesian students are accustomed to solving problems whose level of thinking is limited to knowledge and application (OECD, 2015; 2019; Putri & Zulkardi, 2018)

Efforts are needed to support students in developing their higher order thinking skills. One of the efforts that teachers can do, is by designing mathematics learning using teaching materials that are oriented towards higher order thinking skills. Giving math problems that function as learning material is an alternative way that can be considered. Giving mathematical problems is believed to stimulate students in learning mathematics (Watson & Ohtani, 2015; van Galen & van Eerde, 2018). Furthermore, math problems, using context in students' daily lives, help students to start learning mathematics. By applying context in the math problem, students will be interested to analyze through this context (van Galen & van Eerde, 2018; Rahayu, Putri, & Darmawijoyo, 2018; Meryansumayeka, Zulkardi, & Putri, 2019). Students' understanding of the context in the given math problems will lead students to think mathematically.

There are a lot of studies related to the development of PISA-like mathematical problems (Zulkardi & Kohar, 2018). Several studies have used the development of these questions to describe students' problem-solving abilities (Novita, Zulkardi, & Hartono, 2012), mathematical literacy (Oktiningrum, Zulkardi, & Hartono, 2016), creativity (Novita & Putra, 2016), and higher order thinking skills (Meryansumayeka, Zulkardi, Putri, & Hiltrimartin, 2020). Some have developed questions using local context (Kamaliyah, Zulkardi, & Darmawijoyo, 2013; Charmila, Zulkardi, & Darmawijoyo, 2016; Jannah & Putri, 2019). According to Freudenthal (Sembiring, Hadi, & Dolk, 2008), a phenomenon in everyday life can be used to attract students to learn mathematics. One phenomenon that is currently happening is the Covid-19 pandemic. This pandemic is a problem and affects all aspects of human life including the student learning process (Bakker & Wagner, 2020). However, we can use this phenomenon to attract students to think about this, and come up with solution to this pandemic.

To be able to help students in forming their higher order thinking skills, it is necessary to understand how students work on these math problems. Thus, this study was carried out with the aim of designing PISA-like mathematical problems using Covid-19 context and describing how students work with those problems.

METHOD

This research used design research as a research method. This research consists of three stages, namely the preparation stage, the design stage and the evaluation stage. The evaluation stage includes self-evaluation, expert review, one-to-one, small group, and field test (van den Akker, 2013).

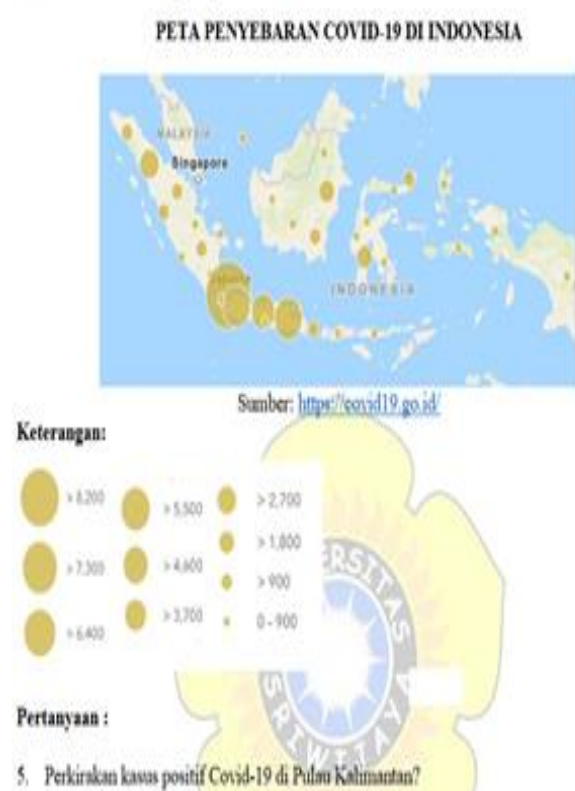
There were 29 high school students in Palembang city were involved. The students are 15 years old and have different level mathematical skills. Data were collected through tests, observations, interviews, and documentations. The test used was in the form of 10 math questions using the context of Covid-19 which were adapted from PISA questions. We used observation to determine students' behavior when they solve questions. We document the students' work to see how students think in solving questions. Meanwhile, we used interviews to clarify students' answers and explore their understanding. All data were analyzed descriptively.

RESULTS AND DISCUSSION

In the preparation stage, researchers analyzed the PISA framework, mathematical topics, and the context of Covid-19. In the next stage, researchers developed 10 PISA math problems using the Covid-19 context. In addition, researchers prepare observation and interview sheets to be used during trials.

After finishing the design of the questions and research instruments, researchers evaluated the mathematical problems designed. In the self-evaluation stage, the mathematical problems design was validated through a Focus Group Discussion. It was attended by several teachers as practitioners of learning mathematics in classrooms and lecturers who had extensive experience in development research. After that, the mathematical problems were corrected based on the input obtained in FGD. Then, the mathematical problems were tried out to students in the one-to-one stage, small groups stage, and field test.

Perhatikan informasi pada gambar berikut.



Translate in English:

Look at the information in the following Picture

COVID-19 SPREAD MAP IN INDONESIA



Note:



The question is:

Estimate the number of positive cases of Covid-19 in Kalimantan island!

Figure 1. A shape and space task using Covid-19 contexts

Figure 1 shows one mathematical-like PISA task designed. This task used the spread of covid-

19 context in Indonesia. The content was about space and shape. In this problem, students were asked to estimate the number of Covid-19 cases happened in Kalimantan. In solving it, the students needed to estimate the total number of positive cases by examining the size of the circles.

5. 0 0 0 0 0 $\Rightarrow$ Kalimantan
 $= 2.700$
 $1.800 = 5.400 + 0 - 900^{(2)}$
 $\begin{array}{r} 900 \\ \times 3 \\ \hline 2.700 \end{array}$
 5.400
 Jadi, perkiraan kasus positif covid-19
 di Pulau Kalimantan $= 4.500 < \text{jumlah}$
 $\text{positif} < 5.500$
 Jawaban : d. 4500 < jumlah positif ~~kasus~~ < 5500

Translate in English:

Therefore, the estimate of Covid-19 positive cases on the Kalimantan island is $4500 < \text{the number of positive cases} < 5500$

The answer is d. $4500 < \text{positif case numbers} < 5500$

Figure 2. Student's answer in solving a mathematical problem using Covid-19 spread context in Kalimantan

Student's answer in Figure 2 shows that there were circles drawing by the student as representation of Covid-19 spread area in Kalimantan. The student also tried to calculate the number of Covid-29 cases in Kalimantan by multiplying 3 same little circles that have range between 0 and 900. It looks like the student used the biggest number that is 900 and multiply it with 3 (3 circles). Then, the student estimated that the number of the biggest circle was 1800 and the number of medium one was 900. All numbers were added up together and the results was 5400. Then, the student referred to the option which 5400 is found within the range, that is between 4500 and 5500. The way students answered was recorded in transcript below.

R : "How did you answer this problem?"

S : "At first, I saw the question. Then I understand that, the question is asking about the number of Covid-19 spread cases in Kalimantan."

R : "Then?"

S : "Then I saw Kalimantan and I observed that there were 5 circles. There were 3 small circles, 1 medium, and 1 big one. "

R : "Ok"

S : "Then I estimated that the small one is around 900, the biggest one is around 1800, and

the medium is more than 900.”

R : “So, you observed the number of small circles, medium circles and big circles?”

S : “Yes”

R : “And then?”

S : “And I added all numbers, and it was 5400”

R : “So, at what range it was?”

S : “D, between 4500 and 5500”

Based on the transcription, student started solving the problem by reading the question directly. Then she looked at the picture. She estimated the values of all circles by referring to information about each circle given. She added up all estimated numbers and chose an answer, in which the number was included in the range.

HAND SANITIZER

Perhatikan gambar berikut.



Sumber: m.cnnindonesia.com

Untuk membuat satu botol *hand sanitizer* standar WHO dibutuhkan bahan - bahan sebagai berikut.

Nama Bahan	Ukuran
Alkohol	840 ml
Hydrogen Peroksida	40 ml
Gliserol	15 ml
Aquadest	60 ml

Pertanyaan :

1. Jika seseorang ingin membuat *hand sanitizer* berstandar WHO dengan bahan – bahan 5.040 ml Alkohol, 124 ml Hydrogen Peroksida, 64 ml Gliserol, dan 320 ml Aquadest, maka ada berapa botol yang dapat dibuatnya?

Translate in English:

HAND SANITIZER

Look at the picture below



Sumber: m.cnnindonesia.com

To make one bottle of WHO standard hand sanitizer, the following ingredients are needed

Ingredient	Amount
Alcohol	840 ml
Hydrogen Peroxide	40 ml
Glycerol	15 ml
Aquadest	60 ml

The question is:

Suppose that one wants to make a WHO standard hand sanitizer. He has 5040 ml of alcohol, 124 ml of Hydrogen Peroxide, 64 ml of Glycerol, and 320 ml of Aquadest. How many bottles can he make?

Figure 3. Quantity task using Covid-19 contexts

The mathematical task in Figure 3 was about quantity. The context is about making hand sanitizer based on WHO standard. The ingredient information is given in the table. The task asks students to calculate the number of hand sanitizer bottles to produce, using the given ingredients.

$$\begin{array}{lcl}
 5040 : 840 & 124 : 40 & \\
 = 6 & = 3 & \\
 64 : 15 & 320 : 60 & \\
 = 4 & = 5 & \\
 3 \text{ sub} = 3 & &
 \end{array}$$

Figure 4. Student's answer in whole numbers

Figure 4 shows that the student used all information given for his calculation. From the information given, there are 4 ingredients of hand sanitizer. The student tried to determine the number of bottles to produce. He divided the number of each ingredient available with the amount of each ingredient needed to make 1 bottle of hand sanitizer. He did not really count the exact number, but he only estimated the nearest whole number.

R : How did you find the answer?

S : I divide the amount of ingredients available by the number of ingredients needed.

R : How did you get 3?

S : 124 divided by 40 is 3 something

R : Do you think the answer is 3? Why not 6?

S : If we make 6 bottles then we cannot use Hydrogen Peroxide. That's not enough. It is just enough for 3 bottles.

From the transcription, the student use estimation in determining the result of division of the number. In answering the question, he had to find the smallest number of the results obtained from his calculation. He had a reason in finding the answer. He argued that the other ingredient was not enough to produce more than 3 bottles of hand sanitizer.

$$\begin{array}{lcl}
 \frac{5040}{840} \Rightarrow 6 & \frac{64}{15} \Rightarrow 4,26 & \\
 \frac{124}{40} \Rightarrow 3,1 & \frac{320}{60} = 5,33 & \\
 \text{maka hand-sanitizer yg dapat dibuat ialah 3 botol} & &
 \end{array}$$

Translate in English:

Therefore, the number of hand sanitizers that can be made is 3 bottles

Figure 5. Student's answer in decimal numbers

Different from Figure 4, the result of student's calculation in Figure 5 was more precise. She used

division and found the result in decimal numbers. However, both student's answer in Figure 4 and Figure 5 have the same conclusion, that the maximum number of hand sanitizer bottles to produce was 3 bottles.

UNIT 5
Perhatikan data pada tabel berikut untuk menjawab soal No. 6 – 7

DATA COVID-19 DI INDONESIA
Update terakhir: 12 September 2020, 15:53 WIB

Di bawah ini merupakan data 10 Provinsi tertinggi Covid-19 di Indonesia.

No.	Provinsi	Terkonfirmasi	Meninggal	Sembuh
1	DKI Jakarta	52840	1386	39793
2	Jawa Timur	37839	2746	29924
3	Jawa Tengah	17460	1139	10969
4	Jawa Barat	14231	294	7493
5	Sulawesi Selatan	13235	376	10103
6	Kalimantan Selatan	9249	386	7319
7	Sumatera Utara	8362	355	5042
8	Bali	7113	168	5593
9	Kalimantan Timur	5752	238	3445
10	Sumatera Selatan	5018	297	3601

Sumber: <https://covid19.go.id/>

Pertanyaan :

6. Berdasarkan tabel tersebut, provinsi manakah yang jumlah pasien Covid-19 dalam perawatan paling tinggi? Berikan alasannya

7. Lingkari Ya atau Tidak untuk setiap pernyataan berikut ini berdasarkan informasi di atas!

Pernyataan	Apakah pernyataan ini benar?
Sumatera Selatan adalah provinsi yang memiliki persentase kesembuhan paling tinggi	Ya / Tidak
Provinsi Jawa Tengah dapat mencontoh cara penanganan Covid-19 dari Provinsi DKI Jakarta	Ya / Tidak

Translate in English:

Look at the data in the following table to answer questions 6 – 7

COVID -19 DATA IN INDONESIA
Last Update: September 12, 2020, 3.53 PM

The following are data on the 10 highest provinces of Covid-19 cases in Indonesia

No	Province	Confirmed	Died	Recovered
1	Jakarta	52840	1386	39793
2	East Java	37839	2746	29924
3	Central Java	17460	1139	10969
4	West Java	14231	294	7493
5	South Sulawesi	13235	376	10103
6	South Kalimantan	9249	386	7319
7	North Sumatra	8362	355	5042
8	Bali	7113	168	5593
9	East Kalimantan	5752	238	3445
10	South Sumatra	5018	297	3601

Questions:

6. Based on the table, which province has the highest number of Covid-19 patients in care? Give your reasons!

7. Circle yes or no for each of the following statements, based on the information above!

Statement	Is the statement true?
South Sumatra is the province with the highest recovery percentage	Yes/ No
Central Java Province can adapt the handling of Covid-19 from the DKI Jakarta province	Yes/No

Figure 6. Uncertainty and data task using Covid-19 context

Figure 6 shows a mathematical task with data provided in a table. One problem was analyzing

which province has the biggest recovery percentage. The other problems were evaluating the statements whether it's True/False.

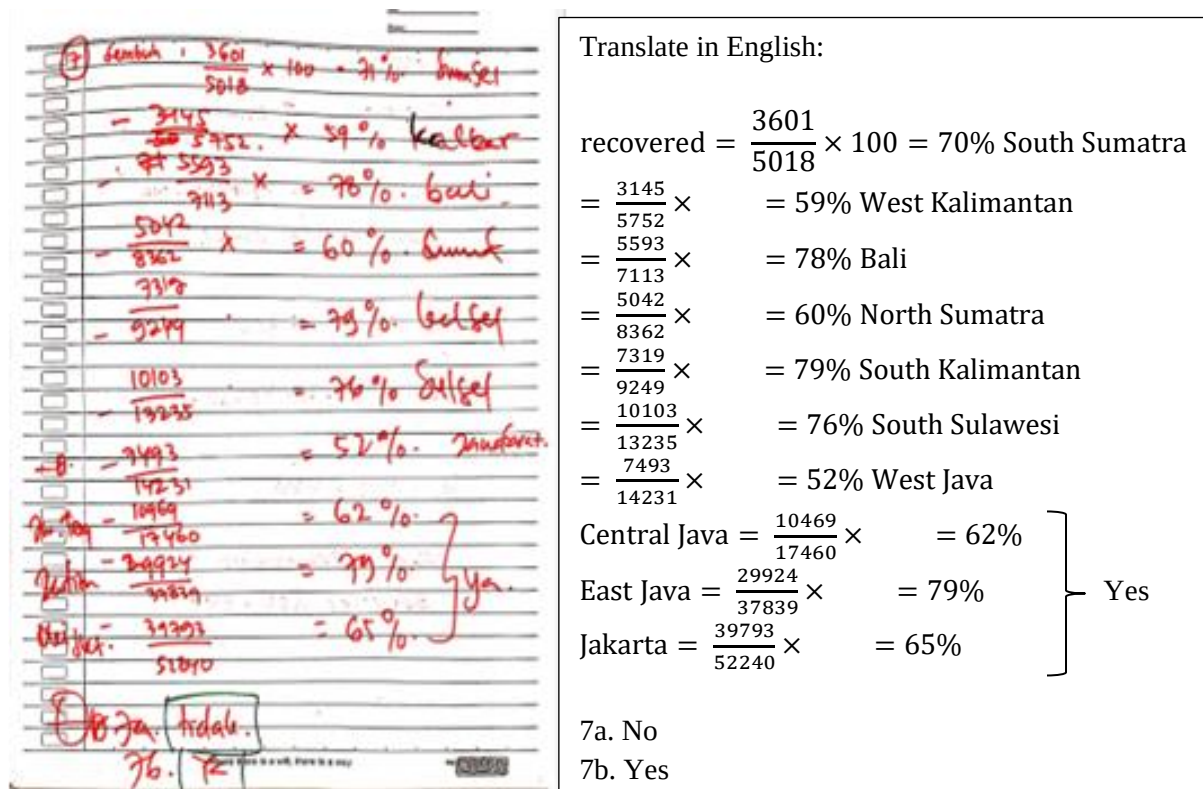


Figure 7. The student calculated all data

In solving the problems in Figure 7, a student tried to calculate the recovery percentage of all provinces. The student did not calculate in exact way, but only by estimating the results to the nearest whole numbers. Then, she looked at the biggest number of the percentage. There, she found that the highest percentage were in East java and South Kalimantan. For the first True/False problems, using her calculation of the data, she concluded that the recovery percentage of South Sumatra was not the highest one. Therefore, she concluded that the statement was false. For the second statement, she concluded that the statement was true since the recovery percentage in Jakarta was higher than in Central Java. However, there were also students did not calculate all data, but they only calculated for some data. The following transcription described the situation.

S: I thought the statement (the first statement of problem no. 7) was false

R: Why?

S: Because Bali's recovery percentage was higher than South Sumatra

R: How did you find that?

S: At first, I divided the number of recoveries with the number of confirmed cases in South Sumatra. Then, I multiplied by 100%. After that, I looked at data in Bali. I also did the same way in determining the recovery percentage in Bali. Evidently, the recovery percentage in Bali

is higher than the South Sumatra.

In the transcription, we see that after the student found the recovery percentage in South Sumatra, she looked at data of the other province and calculated its recovery percentage. When she found out that recovery percentage of the other province was higher than South Sumatra, she concluded that the recovery percentage in South Sumatra was not the highest one. Then, she marked the statement as false.

Based on the results, designing mathematical problems can provoke students thinking mathematically. Van Galen and van Eerde (2018) stated that mathematical problem can stimulate student in learning mathematics. Using context in mathematical problems is also a good way in attracting students to learn mathematics, since they understand the context and can explore mathematics in it from analyzing the problems given (Sembiring et al., 2008; van den Akker, 2013; Meryansumayeka et al., 2019).

Students are encouraged to use their analytical skills in solving PISA-like mathematical problems developed using Covid-19 context, like displaying in Figure 1, 3, and 6. Brookhart (2010) stated that analytical skill is one of higher-order thinking skills. In using this skill, students need to see the relation among information and find out the possible strategies that satisfy the condition.

In solving a PISA - like mathematical problem, students tried to understand it by reading the question. This is the first step of students in solving problems (Nurkaeti, 2018). By reading the question, student understand the problem. However, students did not read every part of the problem. Most of students start by reading the question first. This happened because the students were lazy in reading all of the sentences. This is in line with what Hoogland, Pepin, de Koning, Bakker, and Gravemeijer (2018) stated that a “word” problem, when a task is described in long words, it gives negative effect to students’ performance. In understanding the problem, students also looked at the picture or the table given and refer some information from there. Hoogland, Pepin, de Koning, Bakker, and Gravemeijer (2018) and Hoogland, de Koning, Bakker, Pepin, and Gravemeijer (2018) also stated that using pictures in mathematical problems may help students in understanding the problems. A picture helps student to understand the problem as well as giving information visually (Hoogland, Pepin, de Koning, Bakker, & Gravemeijer, 2018; Hoogland, de Koning, Bakker, Pepin, & Gravemeijer, 2018)

When students saw a problem showing less data, most of them used all information in finding the answer. Since the numbers shown in data were not exactly the same as in the answer option, they referred to all data from the task and then estimate the answer. It also happened when the problem displaying a lot of data, students calculated all data and compared them to find the highest percentage as described in Figure 7. What students did was their strategy in finding the answer; finding the highest percentage, by calculating and comparing among data. After students understood the problem, they used strategy or procedure that they know (Nurkaeti, 2018).

Some students used estimation in finding the result of division, they did not count it in the exact number. The steps they made is shown in Figure 4 and Figure 7. This happened because the students

used multiplication in determining the result of division. They tried to find a number, which multiplication result in the nearest whole number to the solution. This is in line with Dubé and Robinson (2018) that, in determining the result of division of two numbers, students look at the relation to the multiplication that appropriate to those numbers.

CONCLUSION

This study has developed 10 PISA-like mathematical problems using Covid-19 context. In solving those problems, most of student read the question directly, observed and referred information on the picture or the table. For the problems showing less data, students looked at all data and used them to find the solution. It also happened for the problem displaying much data. Some students also used estimation in solving mathematical problems related to division of numbers

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DEDUCTIVE OR INDUCTIVE? PROSPECTIVE TEACHERS' PREFERENCE OF PROOF METHOD ON AN INTERMEDIATE PROOF TASK

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Abstract

The emerging of formal mathematical proof is an essential component in advanced undergraduate mathematics courses. Several colleges have transformed mathematics courses by facilitating undergraduate students to understand formal mathematical language and axiomatic structure. Nevertheless, college students face difficulties when they transition to proof construction in mathematics courses. Therefore, this descriptive-explorative study explores prospective teachers' mathematical proof in the second semester of their studies. There were 240 pre-service mathematics teachers at a state university in Surabaya, Indonesia, determined using the conventional method. Their responses were analyzed using a combination of Miyazaki and Moore methods. This method classified reasoning types (i.e., deductive and inductive) and types of difficulties experienced during the proving. The results conveyed that 62.5% of prospective teachers tended to prefer deductive reasoning, while the rest used inductive reasoning. Only 15.83% of the responses were identified as correct answers, while the other answers included errors on a proof construction. Another result portrayed that most prospective teachers (27.5%) experienced difficulties in using definitions for constructing proofs. This study suggested that the analytical framework of the Miyazaki-Moore method can be employed as a tool to help teachers identify students' proof reasoning types and difficulties in constructing the mathematical proof.

Keywords: deductive-inductive reasoning, proving difficulties, mathematical proof, prospective teachers

Abstrak

Memunculnya bukti matematika formal merupakan komponen penting dalam mata kuliah matematika tingkat lanjut. Beberapa perguruan tinggi telah mengubah mata kuliah matematika dengan memfasilitasi mahasiswa untuk memahami bahasa matematika formal dan struktur aksiomatik. Namun demikian, mahasiswa menghadapi kesulitan ketika mereka beralih ke konstruksi pembuktian dalam mata kuliah matematika. Oleh karena itu, penelitian ini bertujuan untuk mengeksplorasi bukti matematika calon guru di perkuliahan Semester 2. Metode yang digunakan dalam penelitian ini adalah penelitian deskriptif-eksploratif. Partisipan dalam penelitian ini adalah 240 calon guru matematika di sebuah universitas negeri di kota Surabaya, Indonesia. Respons mereka dianalisis menggunakan kombinasi metode Miyazaki-Moore. Metode ini mengklasifikasikan jenis penalaran yang dilakukan, yaitu deduktif dan induktif dan jenis kesulitan yang dialami selama proses pembuktian. Beberapa hasil menunjukkan bahwa 62,5% dari calon guru menggambarkan penalaran deduktif sedangkan sisanya menerapkan penalaran induktif. Selain itu, hanya ada 15,83% dari jawaban yang diidentifikasi sebagai jawaban yang benar, sedangkan jawaban yang lain menunjukkan kesalahan terkait konstruksi bukti. Kami menemukan bahwa sebagian besar respons calon guru (27,5%) mengalami kesulitan dalam menggunakan definisi dalam membangun bukti. Untuk saran, kerangka kerja analitik metode Miyazaki-Moore dapat digunakan sebagai alat yang bermanfaat bagi guru untuk mengidentifikasi tipe-tipe bukti penalaran siswa dan kesulitan dalam membangun bukti matematika.

Kata kunci: penalaran deduktif-induktif, kesulitan dalam pembuktian, bukti matematis, calon guru

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The construction of formal mathematical proof is an important component of advanced mathematics courses for undergraduate degree (Shaker & Berger, 2016). In recent years, some universities have transformed mathematics courses by introducing the transition of proof or introduction to mathematical reasoning courses

(Selden & Selden, 2007; Smith, 2006), which facilitates college students to understand formal mathematical language and axiomatic structure. However, Clark and Lovric (2008) explored challenges faced by college students as they make the transition to proof construction in mathematics courses. This transition requires college students to change their types of reasoning, for instance, shifting the informal language to formal one, reasoning from mathematical definition, understanding and applying the theorem, and making connections between mathematical objects (Clark & Lovric, 2008).

In addition, college students, including prospective teachers, are also demanded to conceive several skills: a) recognizing reasoning and proof as fundamental aspects of mathematics, b) making and investigating allegations of mathematical conjectures, c) developing and evaluating mathematical arguments and proofs, and d) selecting and using different types of reasoning and methods of proof (National Council of Teachers of Mathematics [NCTM], 2000). Blanton, Stylianou, and David, (2003) agreed that college students need to develop required proving skills to construct a proof. In this case, teachers' knowledge about proof must be given to students because that can help the students strengthen the concept and skill of proof (Carrillo, et al, 2018; Stylianides, 2007). Such skills, more particularly, are also necessary for prospective teachers due to the teacher's need for perceiving a deep understanding of nature and the role of proof for conducting instructional practices (Jones, 1997). Moreover, the math teachers' rationales beyond teaching proof and proving in schools are due to the fact that students have experienced similar reasoning to the mathematicians, such as learning a body of mathematical knowledge and gaining insight about why assertions are true. They also can teach students logical thinking, communication, and problem-solving skills in mathematics.

Although proving is an important part of advanced mathematics, many studies indicated that students often have difficulties in constructing a proof (Moore, 1994; Selden, Benkhalti, & Selden, 2014; Selden & Selden, 2007). Epp (2003) reported that a 'poor' mathematical proof process is caused by the lack of proof-writing attempts. In addition, Moore (1994) carried out an observation of some students' transition to college in which most of them stated that they only memorized the proof since they did not understand proof and how to write it. Furthermore, Edwards and Ward (2004) said that the students could not use mathematical definitions or construct the relation between every day and mathematical languages.

In connection with examining student's mathematical proof, Miyazaki and Moore methods might have inspired many researchers for analyzing student's proof with particular objectives. For example, Kögce, Aydin and Yildiz (2010) adopted Miyazaki's (2000) classification of proof to investigate high school students' level of proof based on types of reasoning. Furthermore, Ozdemir and Ovez (2012) looked for the relationship between prospective teachers' perception proof types proposed by Almeida (2000) and their proving processes related to the experienced types of difficulties (Moore, 1994). In relation with students' common error and misconceptions in mathematical proving associated with the use of Moore's error category of proof, Stavrou (2014) found that the students did not necessarily understand the content of relevant definitions or how to apply them in writing proofs. Another study found that students got difficulties in creating definitions that conformed their concept images or

accepted definitions of basic concepts (Dickerson & Pitman, 2016). While those studies concern on a single objective on how students' proof is assessed, the present study highlights that individuals' performance regarding mathematical proof can be explained from at least two aspects namely types of reasoning they involve (Miyazaki's method) and types of difficulties they experience during the proving process (Moore's method). Hence, the researchers argue that obtaining data on both aspects simultaneously leads a broader and more likely fruitful knowledge on how individuals, prospective teachers rather, deal with mathematical proof. Therefore, the present study aims to explore the prospective teachers' proof regarding their types of reasoning and difficulties in a mathematical proof.

Proof is an important aspect in mathematics because it is the main component in understanding mathematics (Kögce et al., 2010) and mathematical thinking (Hanna et al., 2009). Consequently, learning mathematics by mastering mathematical proof along with how to construct it becomes a strategic view (Balacheff, 2010). De Villiers (1990) and Knuth (2002a) stated that the role of mathematical proof is to verify the correctness of a result or truth of a statement, to communicate mathematical knowledge, and to apply an axiomatic system. Its purpose helps investigate the trueness or falseness of an argument regardless the cases and conditions (Baki, 2008) and shows the relevance of the justifications (Lee, 2002).

There are two universally recognized proving methods namely deduction and induction (Kögce et al., 2010; Miyazaki, 2000). Deduction method of proof involves several methods encompassing direct proof, proof by contraposition, and proof by contradiction (Baki, 2008; Morali et al., 2006). Deduction method in mathematics begins with a general statement or hypothesis and examines the possibilities to reach a specific logical conclusion (Morris, 2002). Induction method is generally used by 8th grade students or secondary school students because they have already learned to prove numerical or geometrical proposition (Miyazaki, 2000). These two methods are based on the types of reasoning used by someone in carrying out a proving process, of which each respectively refers to deductive reasoning and inductive reasoning. Deductive reasoning is unique because it is a process of deducing conclusions from known information (premise) based on formal logic rules, where the conclusions must come from information provided and do not need to validate them with experiments (Ayalon & Even, 2008). Whereas, Christou and Papageorgiou (2007) conveyed that inductive reasoning is a reasoning process from specific premises or observations to reach a general conclusion or an overall rule. Of those two, deductive reasoning, which is used in a deductive proof, is considered the preferred tool in many mathematical communities to verify mathematical statements and demonstrate universality. Therefore, Ayalon and Even (2008) argued that deductive reasoning is often used as a synonym for mathematical thinking.

Knowledge of mathematical proof is considered as one essential component of subject matters (Shulman, 1986) in which mathematics teachers must acquire. Jones (1997) argued that teachers will have an extremely secured subject knowledge base of mathematical proof if they teach it accurately and confidently.

Most researchers who have examined teachers' knowledge of proof have centered on teachers' acceptance of empirical versus deductive arguments as valid proofs. Knuth (2002b) investigated sixteen in-service secondary school mathematics teachers' knowledge about what constituted a proof. Martin and Harel (1989) assessed the notions of proof performed by 101 pre-service elementary school teachers by giving their participants some statements accompanied by predetermined arguments and asking them to rate for revealing the validity. Both Knuth (2002b) and Martin and Harel (1989) concluded that most teachers correctly identified a valid argument and also wrongly accepted invalid arguments as proofs. Some pre-service elementary teachers accepted empirical arguments as proofs (Martin & Harel, 1989; Morselli, 2006; Simon & Blume, 1996).

Teacher's knowledge about proof is indeed not limited to understanding how to construct valid proofs. More broadly, it is related to the knowledge of both content and pedagogical aspects of proof. Steele and Rogers (2012) propose the so-called 'Mathematical knowledge for teaching proof' that can be considered as a meaningful framework to assess teacher's knowledge about proof. Steele and Roger (2012) also mention components of proof knowledge comprising knowledge of defining proof, identifying proofs and non-proofs, creating mathematical proofs, and understanding the roles of proof in mathematics. The first three components give main features on the content knowledge of mathematical proof, while the last component gives attention about the work of teaching proof since it is related to, for instance, checking or confirming students' thinking on the truth of a known idea, unpacking students' thinking and reasoning beyond their decision why a statement is true, confirming students' conjectures, and developing students' new mathematical ideas.

This study focuses on one of the teacher's knowledge, particularly on studying prospective teachers' proof identification and creating mathematical proofs through recognizing them across representations in case of two types of reasoning namely deductive and inductive reasoning and types of difficulties during a proving process.

METHOD

This study used descriptive-explorative research design to explore prospective teachers' types of reasoning and difficulties in carrying out a mathematical proving process. There were 240 prospective teachers who studied at Department of Mathematics of a state university in Surabaya, Indonesia, as the research participants. They were in the first semester so that the present study used convenience sampling method (Miles & Huberman, 1994). The participants consisted of 216 female and 24 male students (aged around 18 years old in average). The data were collected in three years (from 2013 to 2015) by providing initial tests to the participants, as many as 80 prospective teachers per academic year. There were 2 classes per academic year, in which each class consisted of 40 prospective teachers.

Even though Moore's (1994) research used interview to see errors in mathematical proof, the present study tended to examine more on deductive and inductive proof without employing interview like what Miyazaki (2000) did. The data were collected using a simple task of constructing one

mathematical proof "*Prove that the sum of two odd numbers is an even number*". Actually, the task type could be more than one, such as *the sum of two even numbers is even, the sum of odd and even numbers is odd, or the subtraction variation from two even, odd, or even-odd numbers*. However, the main point of this study was a proof method whether using deductive common symbols or certain numbers that tended to be inductive. This, therefore, only required one sufficient problem determined to represent the use of deductive and inductive reasoning. This simple task consisted of a question that was similar to the task used by Özer and Arökan (2002) in constructing proof in accordance with the types of reasoning. Instead of only types of reasoning, the present study's analysis also concerned on exploring prospective teachers' proof based on the types of difficulties. Moreover, this task was selected since it was often found in Indonesian secondary school curriculum, which had been frequently learned by prospective teachers who studied early mathematical proof. The process of data collection was carried out during the beginning of number theory and elementary algebra courses in their study. Each prospective teacher had been given 15 minutes to complete the task. Afterwards, each prospective teacher's responses were assessed to investigate the prospective teacher's comprehension of their deductive and inductive knowledge in constructing a proof along with their difficulties in constructing their proofs.

Table 1. Moore's types of difficulties in performing mathematical proof (Moore, 1994)

Type of difficulties	Discussion
D1	Prospective teachers did not know the definitions. That is, they were unable to state the definitions.
D2	Prospective teachers had a little intuitive understanding of the concepts.
D3	Prospective teachers' concept images were inadequate for doing the proofs.
D4	Prospective teachers were unable, or unwilling, to generate and use their own examples.
D5	Prospective teachers did not know how to use definitions to obtain the overall structure of proofs.
D6	Prospective teachers were unable to understand and use mathematical language and notation.
D7	Prospective teachers did not know how to start making a proof.

The data of participants' responses about proof correctness were analyzed by using Miyazaki's (2000) classification for types of reasoning in mathematical proof and Moore's (1994) classification

for types of errors in a mathematical proof. Table 1 and Table 2 show the scheme of proof analysis used to analyze prospective teachers' proof.

Table 2. Miyazaki's types of reasoning in performing mathematical proof (Miyazaki, 2000)

Representation	Contents	
	Inductive reasoning	Deductive reasoning
Functional language used for demonstration	Proof D	Proof A
Other languages, drawings, and/or manipulated objects	Proof C	Proof B

Proof A was the type of proof when deductive reasoning was involved and a functional language was used in the course of making a proof. Proof B was the type of proof where deductive reasoning was involved and other languages, drawings, and movable objects were used in the course of making a proof. Proof C was the type of proof where inductive reasoning was involved and other languages, drawings, and movable objects were used. Proof D was the type of proof where inductive reasoning was involved and a functional language was used.

Table 3. A combined classification of Moore-Miyazaki category of proof

Classification								
Miyazaki (Type of reasoning)			Moore (Type of difficulties)					
Type	Correct Proof	Conceptual understanding					Mathematical languages and notations	Getting started on a proof
		D1	D2	D3	D4	D5	D6	D7
Proof A	%	AD1	AD2	AD3	AD4	AD5	AD6	AD7
Proof B	%	BD1	BD2	BD3	BD4	BD5	BD6	BD7
Proof C	%	CD1	CD2	CD3	CD4	CD5	CD6	CD7
Proof D	%	DD1	DD2	DD3	DD4	DD5	DD6	DD7

In combining the entries presented in Table 1 and Table 2, the present study applied the coding category presented in Table 3 for conducting an analysis. For example, since Proof A referred to a proof involving deductive reasoning with functional language used for demonstration, and D1 referred to a proof difficulty indicated by prospective teachers' misconception on the definition along with their inability to state the definition, then AD1 referred to a proof performance involving deductive reasoning

that indicated difficulties shown by the inability to state definitions. Furthermore, the column correct proof portrays that percentage of the students' correct proofs.

In addition, the coding was carried out to all the prospective teachers' answers. Since there was more than one possibility of coding given to each answer with different categories, the present study selected the most significant feature of the response category that emerged from the answer. Henceforth, each answer only had one code of category. The coding was carried out by the first author and the reliability of the coding was checked through additional coding by an external coder, who was a teacher educator in our university. It was done based on 20 % of 240 prospective teachers' responses in problem proof. 20% of the population chosen randomly became the minimum sample size used in this study that were determined by using Slovin formula with a 10% error margin. In agreement with the multiple coding procedures, this study calculated the inter-rater reliability for each type, which resulted in Cohen's Kappa of 0.69, indicating that the coding was a substantial agreement (Landis & Koch, 1977).

RESULTS AND DISCUSSION

In this section, the data obtained from the participants were analyzed, discussed, and then presented in Table 4. In accordance with Table 4, there were 38 prospective teachers who were correct in mathematical proof, whereas, the others were wrong in mathematical proof. Table 4 also depicts that 61.66% of the prospective teachers performed Proof A in which this proof required deductive reasoning and functional language used to construct proofs. Meanwhile, 0.84% of the prospective teachers conveyed Proof B with deductive reasoning and manipulated objects or using a sentence without functional language in proof. 31.25% of the prospective teachers showed Proof C in which they used inductive reasoning and other languages, images, and manipulated objects to construct proofs. Moreover, 6.25% of the prospective teachers showed Proof D in which they used inductive reasoning and functional language for constructing proofs. Regarding the correctness of the prospective teacher's responses, the present study found that 15.8% of the prospective teachers' responses were correct and 84.2% of the prospective teachers still experienced difficulties in constructing the proof task. Figure 1 to Figure 15 explain the examples of the results of prospective teachers' proof based on Proof A, Proof B, Proof C, and Proof D.

Proof A

The results showed that Proof A was performed by as many as 15.83% of the prospective teachers, meaning that they worked on the proof task correctly according to the deductive reasoning and functional language in constructing proofs. Meanwhile, 45.83% of the prospective teachers had difficulties in proving caused by several things encompassing less understanding of the concept involved (42.08%), lack of knowledge related to mathematical notations (0.42%), and being stuck in starting the proving process (3.33%). In connection with the less understanding of the concept, the prospective teachers' responses consisted of AD1, AD2, AD3, AD4, and AD5 types, while the

difficulties to get started on proving included AD7 type, and the lack of knowledge about mathematical notation and logic included AD6 type. Figure 1 shows the correct examples of a prospective teacher's answer to Proof A in constructing the proof.

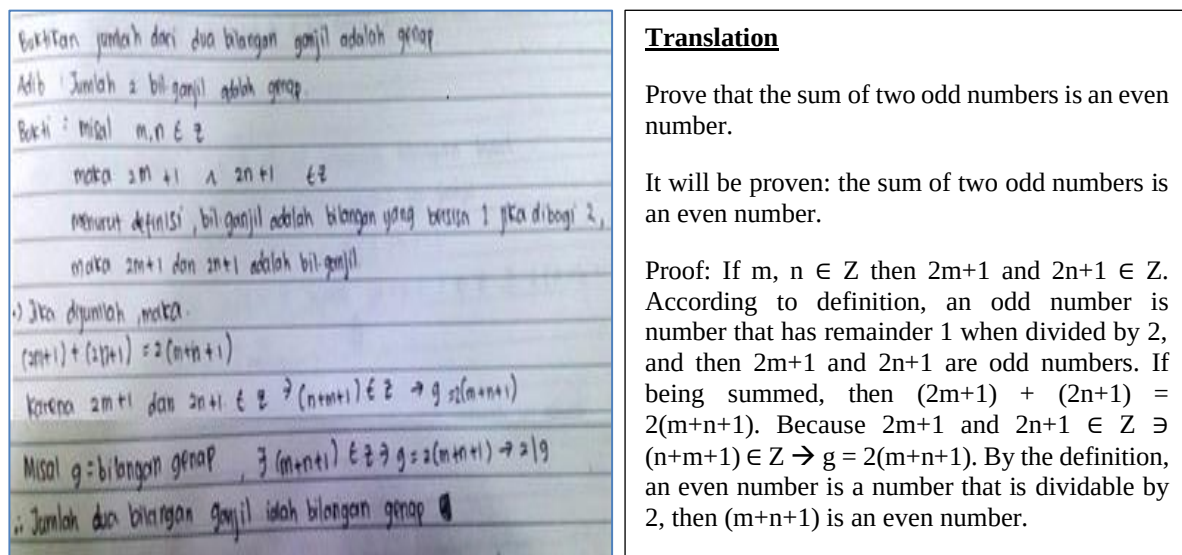


Figure 1. Example of Proof A

Figure 1 shows that prospective teachers worked on proving with the correct response toward the question given. The correct response in this proof, including Proof A, used deductive reasoning and functional language that could be seen in the student work sample (see Figure 1). The deductive reasoning was indicated by the prospective teacher's idea in firstly letting an even number and an odd number with different symbols, which indicated his understanding of the rigorous symbol that had a significant step for being manipulated in the subsequent proving process. Moreover, it also showed some functional languages precisely, such as the symbols of $\in$, $\mathbb{Z}$, $\exists$ and $|$ that indicated their proficiency in dealing with mathematical symbols.

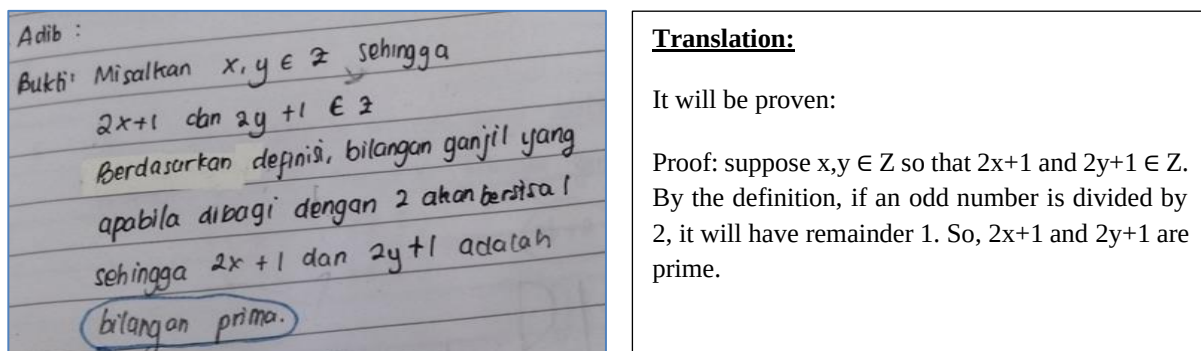


Figure 2. Example of Proof AD1

Figure 2 explains that the prospective teacher's concept in constructing a proof was not well understood, it could be seen that he could not state the definition correctly. It also indicated in the definition of $2x + 1$ and $2y + 1$ when he wrote as "prime number". Whereas, based on the definition,

the form of ' $2x + 1$ ' and ' $2y + 1$ ' were an odd number. By definition, an odd number was a number that had a remainder of 1 when divided by 2 so $2x + 1$ and $2y + 1$ were an odd number. Based on the data, 6.25% of the total prospective teachers using such typical proving process.

Translation:

Odd number + odd number = even number

Suppose: even number = P

odd number = Q

$$P + P = Q$$

$$2P = Q$$

$$P = \frac{Q}{2}$$

Odd number + odd number = even number

$$\frac{Q}{2} + \frac{Q}{2} = Q$$

$$\frac{2Q}{2} = Q$$

Figure 3. Example of Proof AD2

Figure 3 shows that the prospective teacher was not able to construct proofs because of less understanding of the theorem or the concepts involved. The concept of proof that should be proved was used in the proof. The result of $\frac{Q}{2} + \frac{Q}{2} = Q$, which $\frac{Q}{2}$ should have been proved because it was an even number and it was used in the proof. This happened due to prospective teacher's weak intuitive understanding before starting the proving process, so that, she could not solve proof task formally, logically, and relevantly to the definition. Based on the data, 3.75% of the 240 prospective teachers experienced such proving errors.

Translation

Prove that the sum of two odd numbers is an even number.

Answer: suppose odd number = a

$$a + a + 2 = a + 1 \rightarrow \text{an even number}$$

$$2a + 2 = a + 1$$

$$a = 1$$

Thus, the sum of two odd numbers is an even number.

Figure 4. Example of Proof AD3

Table 4. The classification of prospective teachers' answers toward the proof task

Classification																		
Miyazaki (Types of reasoning)			Moore (Types of difficulties)															
Type	Correct Proof		Conceptual understanding										Mathematical language and notation		Getting started on a proof		Total	
			D1		D2		D3		D4		D5							
	n	%	n	%	n	%	n	%	n	%	n	%	n	%	n	%		
Proof A	38	15.8	15	6.25	9	3.75	6	2.5	5	2.08	66	27.5	1	0.42	8	3.33	148	61.66
Proof B	0	0	1	0.42	0	0	0	0	0	0	0	0	1	0.42	0	0	2	0.84
Proof C	0	0	0	0	6	2.5	6	2.5	63	26.25	0	0	0	0	0	0	86	31.25
Proof D	0	0	12	5	0	0	0	0	0	0	3	1.25	0	0	0	0	4	6.25
Number (n)	38	15.8	28	11.6	15	6.25	12	5	68	28.33	69	28.75	2	0.84	8	3.33	240	100

Figure 4 explained that the prospective teacher's understanding of the concepts in constructing proof was unrevealed, especially on his concept images. The language required to express mathematical ideas in the proof was still insufficient and unclear. He began to construct a proof by letting an odd number as a and then being manipulated in an equation resulting $a=1$. However, it was not clear to bring the proof into corresponding directions of a valid proof that was *the sum of two odd numbers is an even number*. Based on the data, 2.5% of the 240 prospective teachers experienced such proving errors.

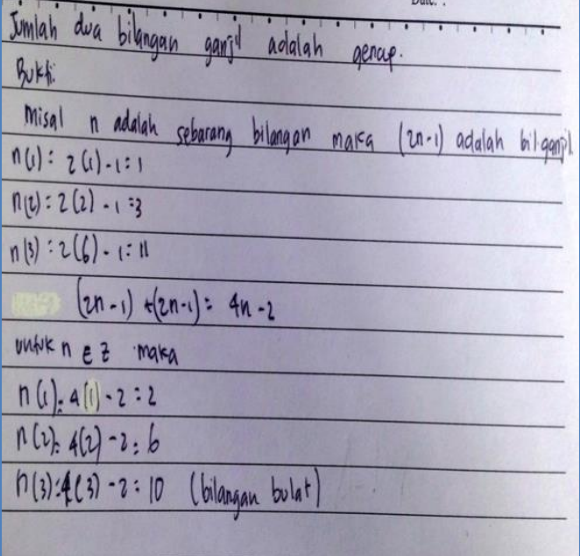
	Translation Prove that the sum of two odd numbers is an even number. Proof: Suppose n is an arbitrary number, then $(2n-1)$ is an odd number. $n(1) = 2(1) - 1 = 1$ $n(2) = 2(2) - 1 = 3$ $n(3) = 2(6) - 1 = 11$ $(2n-1) + (2n-1) = 4n-2$ For $n \in \mathbb{Z}$ then $n(1) = 4(1) - 2 = 2$; $n(2) = 4(2) - 2 = 6$; $n(3) = 4(3) - 2 = 10$ (integer).
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Figure 5. Example of Proof AD4

Figure 5 shows that the prospective teacher started to use functional language that was supposed as n , but it was not completed. The results show that the prospective teacher used several examples to understand the concept of constructing a proof. Moreover, he was still unable to build their examples, which were used to construct proofs. Based on the data, 2.08% of the 240 prospective teachers experienced such proving errors.

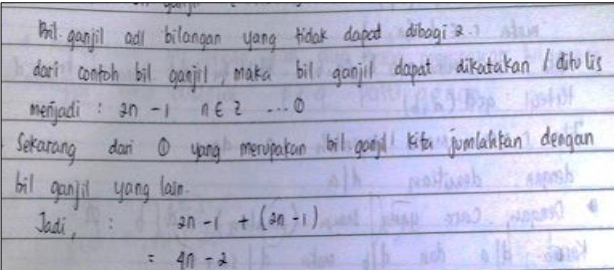
	Translation Odd number is a number that cannot be divided by 2. Based on this definition, then an odd number can be said or written as $2n-1$, $n \in \mathbb{Z} \dots (1)$ From (1), we will add with the order odd number. Thus, $2n-1 + (2n-1) = 4n-2$.
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Figure 6. Example of Proof AD5

Figure 6 shows that the prospective teacher could involve deductive reasoning with a definition, but she still did not know how to use the definition of an odd number correctly. This was indicated by the use of a definition of an odd number namely $2n-1$. Afterwards, she added the other odd number that resulted $(2n-1) + (2n-1)$. Based on the sum of these numbers, she used the same variables namely

n, however, letting two arbitrary odd numbers with the same variables was not accepted due to the possibility that those two numbers could be different. Based on the data, 27.5% of the total prospective teachers experienced such proving errors.

	<u>Translation</u> Suppose A = odd number ($A \bmod 2 = 1$) B = odd number ($B \bmod 2 = 1$) C = even number ($C \bmod 2 = 0$) $A + B = C$ $(A \bmod 2 + B \bmod 2) \bmod 2 = C \bmod 2$ $(1 + 1) \bmod 2 = 0$ $2 \bmod 2 = 0$ $0 = 0$ Thus, $A + B = C$ We can conclude that the sum of two odd numbers is an even number.
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Figure 7. Example of Proof AD6

Figure 7 explains that deductive reasoning was correctly used by the prospective teacher's proof. However, the prospective teacher used particular language and mathematical notation incorrectly. This problem could be seen from the notation "=", which meant "is equal" instead of "is equivalent". The symbol was normally used in congruence involving modulo. Therefore, it could be concluded that the prospective teacher did not fully understand the meaning of a notation "=". Based on the data, 0.42% of the total prospective teachers experienced such proving errors.

	<u>Translation</u> $x + y = Z, Z \in \text{even number.}$
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Figure 8. Example of Proof AD7

Figure 8 shows that the prospective teacher involved deductive reasoning in constructing the proof. The finding $x + y = z$ showed that the prospective teacher used functional language at the beginning of the proof. However, this equation was meaningless due to the poor mathematical argument and interpretation of the symbols given. The prospective teacher in this proof likely had less knowledge of proving so it was difficult to start constructing a proof. Based on the data, 3.33% of the total prospective teachers experienced such typical proving errors.

Proof B

In Proof B, the prospective teachers involved deductive reasoning and other languages, drawings, and movable objects during constructing a proof. In this category, 0.84% of the prospective teachers had difficulties in constructing a proof caused by several things, comprising a less understanding of the concept (0.42%) and less understanding of mathematical notations and language in constructing a proof (0.42%). The following examples show the results of prospective teachers' answers that contained errors in constructing Proof B.

	Translation Odd + odd = even 2 Odd = even Odd = $\frac{\text{even}}{2}$ $\frac{\text{even}}{2} + \frac{\text{even}}{2} = \text{even}$ $\frac{2 \text{ even}}{2} = \text{even}$ even = even
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Figure 9. Example of Proof BD1

Figure 9 explains that the prospective teacher involved deductive reasoning but did not use functional language in constructing a proof. It could be seen from his work showing that “odd + odd = even”. He also could not state the definition correctly as indicated in his definition that “odd = $\frac{\text{even}}{2}$ ”. Whereas, when an even number was divided by 2, the result was also an even number. Hence, he performed the proof incorrectly. Based on the data, 0.42% of the total prospective teachers experienced such proving errors.

	Translation Prove that the sum of two odd numbers is an even number. Answer: $3 + 3 = 6$, where 3 is an odd number, and 6 is an even number. Every odd number is even number + 1. Thus, (even number + 1) + (even number + 1) = even number, cause $1 + 1 = 2$, and 2 is even number. → Even number + even number + 2 = even number. Thus, the sum of odd numbers is an even number.
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Figure 10. Example of Proof BD6

In accordance with Figure 10, it was indicated that the prospective teacher already involved deductive reasoning but she demonstrated the proof by her language without the use of appropriate functional language in constructing a proof. This could be seen from her sentence *every odd number is an even number plus one*. This sentence should employ some symbols using a functional language, for example, $2n + 1$ for an odd number with n integer. Thus, the prospective teacher still did not understand how to use the symbolic language in proof. It could be a result of the limitations of her conceptual understanding about the nature of proof. Based on the data, 0.42% of the total prospective teachers experienced such typical proving errors.

Proof C

In this type of proof, the prospective teachers were not able to prove using inductive reasoning and other languages, drawings, and movable objects. However, 31.25% of the prospective teachers got difficulties in constructing this type of proof caused by less understanding of mathematical concepts. The following example shows student's answer that had difficulties in constructing Proof C.

	Translation Odd number = 1, 3, 5, 7, 9,... $U_n = U_1 + (n-1)b$ Proof: $U_1 + U_2 = 1 + 3 = 4$ $= (U_3 - 1)$ $U_1 + U_2 = (U_3 - 1)$ $U_{(n-1)} + U_n = (U_{n+1} - 1)$ Thus, it is proven that the sum of two odd numbers is an even number.
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Figure 11. Example of Proof CD2

Figure 11 shows that the prospective teacher tried to perform inductive reasoning by giving some examples of the number involved in an arithmetic equation at the beginning of stating a proof. Nevertheless, it was unclear that the concept of proof used in constructing a proof was well presented. For example, from the equation $U_1 + U_2 = 1 + 3 = 4 = (U_3 - 1)$, the prospective teacher concluded that $U_{(n-1)} + U_2 = (U_n + 1 - 1)$. In this case, the prospective teacher still did not understand the whole direction of the proof due to the lack of an intuitive understanding about how a mathematical proof should work. Therefore, he could not finish constructing the proof correctly. Based on the data, 2.5% of the total prospective teachers experienced such typical proving errors.

Contoh $1 + 1 = 2$ $1 + 3 = 4$ $3 + 5 = 8$ Berdasarkan contoh tersebut dibuktikan jumlah bilangan ganjil adalah genap $n = 1$ $\begin{array}{ccccccc} n, & n+1, & n+2, & n+3, & n+4 \\ 1 & 2 & 3 & 4 & 5 \\ \downarrow & & \downarrow & & \downarrow \\ n+1+1 & n+1+2 & & n+1+3 & \end{array}$ Anggota bilangan ganjil (1, 3, 5, 7, ...) $n = 2n+1$	Translation $1 + 1 = 2 \quad 1 + 3 = 4 \quad 3 + 5 = 8$ Based on this example, it is proven that the sum of two odd numbers is an even number. $n = 1$ $n \rightarrow 1, \quad n+1 \rightarrow 2, \quad n+2 \rightarrow 3 \rightarrow n+1+1,$ $n+3 \rightarrow 4 \rightarrow n+1+2, \quad n+4 \rightarrow 5 \rightarrow n+1+3$ The members of odd number: (1, 3, 5, 7, ...) $n \quad 2n+1$
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Figure 12. Example of Proof CD3

Figure 12 depicts the prospective teacher performed inductive reasoning by giving an example of numbers. However, it was noted that her understanding of the concept of proof was still unclear. The prospective teacher started from stating the proof by giving some examples, then generalized them into the formal form. Afterwards, she continued to resume her work by providing other examples. It was incompatible with the concept of inductive proof where the valid examples satisfying the condition of a statement should be generalized into a formal conclusion. Based on the data, 2.5% of the total prospective teachers experienced such typical proving errors.

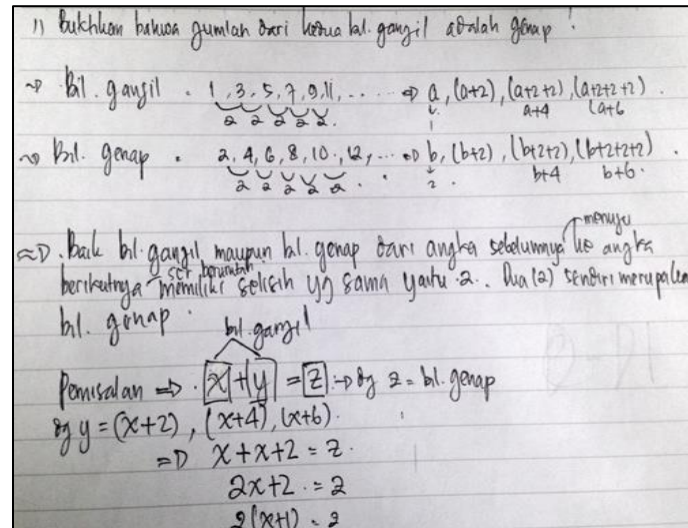
buktikan jumlah dari dua bilangan ganjil adalah genap. Misal: $3 + 3 = 6$. 3 merupakan bilangan ganjil dan hasil penjumlahan dari 3 dan 3 adalah 6. dan 6 merupakan bilangan bulat. dari permasalahan diatas bahwa bilangan ganjil ditambah bilangan ganjil adalah genap	Translation Prove that the sum of two odd numbers is an even number. Suppose: $3 + 3 = 6$ Where, 3 is an odd number and the result of $3 + 3 = 6$, 6 is an integer. Therefore, the sum of two odd numbers is an even number.
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Figure 13. Example of Proof CD4

Figure 13 portrays that the prospective teacher used examples to construct an inductive proof but she did not complete the proof. She was more inclined to mention some numbers on an arithmetical operation, namely $3 + 3 = 6$, but did not conclude her examples of proving to the general form, which was a functional language used in constructing a proof. Based on the data, 26.25 % of the total prospective teachers experienced such proving errors.

Proof D

In relation with Proof D, the prospective teachers could not perform the proving process correctly based on inductive reasoning and functional language in constructing a proof. The number of prospective teachers who got proving difficulties in this category was 6.25%, in which the problem was caused by a weak understanding of the concept in the proof task. The following examples show the prospective teacher's answers that had difficulties in constructing Proof D.

**Translation**

Prove that the sum of two odd numbers is an even number.

Odd numbers = 1, 3, 5, 7, 9, 11, ...

$$\rightarrow = a, (a + 2), (a+2+2), (a+2+2+2) \\ = a, a + 2, a + 4, a + 6$$

Even numbers = 2, 4, 6, 8, 10, 12, ...

$$\rightarrow = b, (b + 2), (b+2+2), (b+2+2+2)$$

$$= b, b + 2, b + 4, b + 6$$

For either odd or even numbers, the difference between every two consecutive number is 2. And, 2 is an even number. Where x, y are odd numbers and z is an even number.

With y = (x+2), (x+4), (x+6)

$$x + x + 2 = z; 2x + 2 = z; 2(x+1) = z$$

Figure 14. Example of Proof DD1

Figure 14 shows that the prospective teacher did not understand how to define an odd number in the form of functional language. It could be seen from the student's answer on the definition of even and odd numbers. The prospective teacher gave a series of example on how odd numbers and even numbers are illustrated as evidence that he started proving inductively. Despite he tried to arrange the general form of the examples, which was the series $a, (a + 2), (a+2+2), (a+2+2+2)$ for even number,

and b , $(b + 2)$, $(b+2+2)$, $(b+2+2+2)$ for odd number. However, it did not show how an odd number and an even number should be mathematically symbolized. Thus, his final step, which was $2(x+1) = z$ yielded a variety of interpretations, did not certainly describe the condition expected in the proof task. Based on the data, 5 % of the total prospective teachers experienced such proving errors.

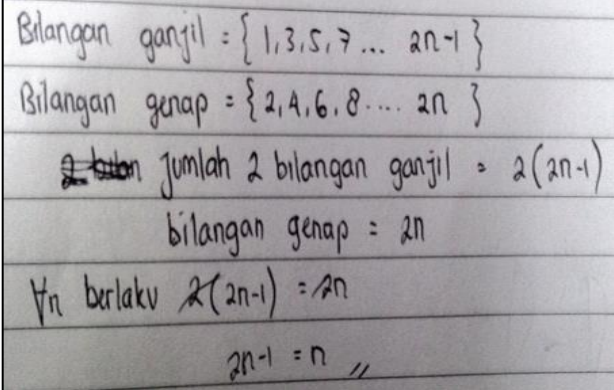
 Bilangan ganjil = $\{1, 3, 5, 7 \dots 2n-1\}$ Bilangan genap = $\{2, 4, 6, 8 \dots 2n\}$ ~~2 kali~~ jumlah 2 bilangan ganjil = $2(2n-1)$ bilangan genap = $2n$ $\forall n$ berlaku $2(2n-1) = 2n$ $2n-1 = n //$	Translation Odd numbers = $\{1, 3, 5, 7, \dots 2n-1\}$ Even numbers = $\{2, 4, 6, 8, \dots 2n\}$ The sum of two odd numbers = $2(2n-1)$ Even numbers = $2n$ $\forall n$ apply $2(2n-1) = 2n$ $2n-1 = n$
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Figure 15. Example of Proof DD5

Figure 15 portrays the prospective teacher used inductive reasoning but he still did not know how to use the definition correctly. The figure also shows $(2n-1) = n$ that indicated that he did not use the definition of odd number correctly. He canceled number 2 on the right and left instead of using the definition of an odd and even number that he had written. Based on the data, 1.25% of the total prospective teachers experienced such proving errors.

Our finding indicates that the prospective teachers apply deductive and inductive methods in constructing a proof. The deductive method consists of two types namely Proof A and Proof B and the inductive method consists of two types covering Proof C and Proof D. Table 4 points out that 62.5% of the prospective teachers use deductive method while 37.5% of them use inductive method. Meaning that, more than half of the prospective teachers' answers use deductive method. Despite some of the prospective teachers having errors in constructing a proof, they already try to construct a proof with deductive and inductive methods. This finding is consistent with the research conducted by Miyazaki (2000) that most students in his study use a deductive method instead of the inductive one in constructing a proof.

Furthermore, the prospective teachers perform Proof A, Proof B, Proof C, and Proof D types with the percentage of 61.66%, 0.84%, 31.25%, and 6.25%, respectively. Therefore, it shows that Proof A is the most commonly found in the prospective teachers' answers than those of other types. When compared to the findings undertaken by Kögce et al. (2010), it does not align with the results of Kögce et al. (2010), in which the study result reports that Proof C is performed by most students than the other types of proof (51.2%). The fact that our study has found many deductive methods in our participants' answers might occur because the proof task given in the present study demands a solver to use a deductive method instead of an inductive one. However, in connection with the results of the

prospective teacher's answers, there are some answers indicating an inductive method, all of which are still incorrect. This finding very likely corresponds to Demiray and Bostan (2017) who report that most of the students' incomplete proof yielded incorrect proofs are caused by the unsuitable inductive method the students use in constructing their proofs. Therefore, it indicates that the type of proof task affects the selection of proving methods.

The results of the present study are also consistent with Miyazaki's (2000) study, where Proof A is performed by the highest number of prospective teachers. The results show that prospective teachers can use deductive reasoning and sufficient techniques of proof, however, there are still some errors in constructing a proof. In addition, Miyazaki (2000) points out that Proof C is performed by the least number of prospective teachers. This does not align with the present study that reveals the fact that Proof C is performed by the second-highest number of prospective teachers. This result shows that the prospective teachers still involve inductive reasoning with other languages, drawings, and objects used in the process of constructing a proof. In this regard, promoting deductive argumentation among students in mathematics education is important since many prospective teachers, including those who enrolled postgraduate study program in mathematics education, often perform logically disconnected premises and conclusions drawn within a mathematical proof (Ndemo, 2019).

Regarding the difficulties in proving, the results show that AD5 type is performed by 27.5% of the prospective teachers, which becomes the highest result of proof. Similarly, Edwards and Ward (2004) convey that many prospective teachers cannot use the definition to make mathematical proofs. Aligned with Moore's category, the use of definition is indeed being one of the difficulties in constructing a mathematical proof. In addition, the present study shows that the number of prospective teachers performing incorrect proofs is bigger than those who make the correct ones. That is, most prospective teachers still experience difficulties in constructing a mathematical proof. 79.98% of the prospective teachers are still weak in understanding the concept of proof, 3.33% of them lack of knowledge, and 0.84% of them get limited ability to construct a proof. This study is consistent with Chin and Lin's (2009) study revealing that most prospective teachers have problems in constructing valid algebraic proofs.

The present study not only indicate the performance of prospective teachers regarding types of reasoning and difficulties in proving processes, but also show the potential use of analytical framework of assessing individuals' proving performance. This framework is developed in order to get a broader insight on how to evaluate types of proof and proving difficulties from a written response representing individual proof performance. It is expected that this analytical framework will complement other frameworks in assessing individual cognitive performance related to proof, such as mathematical knowledge for teaching proof (Buchbinder & McCrone, 2020) as the knowledge of different types of proofs becomes one of vital components in this framework. However, the combined Miyazaki-Moore method used in the framework is carried out to only assess students' written responses. In future, it is

more beneficial if the data include interview results to confirm the detailed information about the students' difficulties in constructing proofs.

CONCLUSION

The most prospective teachers construct a proof by employing deductive reasoning rather than inductive reasoning. It can be seen in Proof A that has been performed by the highest number of prospective teachers. However, some prospective teachers still experience difficulties in constructing a mathematical proof. The types of difficulties mostly found in the prospective teachers' answers include the fact that they cannot appropriately use the definition in making mathematical proofs.

This study only presents the prospective teachers' responses in constructing a proof because the researchers want to know the trend of mapping models in assessing prospective teachers regarding their knowledge about proof constructions. They also have empirically proven that the framework proposed in this study can work. The advantages of using the framework cover the ability to assess students' types of reasoning and difficulties in constructing proofs simultaneously.

As the constructive feedbacks, the framework can be used as an evaluation tool for the needs of mathematics teacher education program in a university curriculum. For further studies, the present study offers a potentially broader insight on assessing learners' cognitive processes to study learners' reasoning process in mathematical proof regarding proving difficulties and types of reasoning since the framework developed in this study has not covered such issue yet. In addition, this framework is intended to only code the responses based on the participants, meaning that every single response can only get chance to be coded in one category of proof based on types of reasoning and difficulties. Thus, it is suggested that the framework can be developed into covering more than one category of proof since, for example, a response from another mathematical proof task may be categorized in more than one type of difficulties.

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LEARNING GEOMETRY AND VALUES FROM PATTERNS: ETHNOMATHEMATICS ON THE BATIK PATTERNS OF YOGYAKARTA, INDONESIA

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Abstract

In general, many people still view mathematics as a subject that is far from reality and culture in everyday life. Historically, in fact, mathematics is very close to daily life and was developed by humans in response to the surrounding phenomena. Indonesia has diverse cultures, including in Yogyakarta. This culture can be used to explore mathematical concepts as a transformational effort to bring mathematics closer to the reality and perception of its people. Besides, we can use culture as the basis of learning mathematics in schools. Therefore, this study seeks to explore a mathematical concept of geometry transformation in the Yogyakarta batik pattern. This is an ethnography study. The research data were collected through observations, literature studies, and interviews with the batik culture practitioner and artist to understand the batik techniques and moral, historical, and philosophical values in each batik motif. This study's results indicate that in Yogyakarta batik, it uses the concept of geometry transformation in the making of Yogyakarta's unique Batik motif. Besides that, each motif or pattern also contains local values. These, namely moral, historical, and philosophical values, can be felt, reflected, and applied in daily life, such as values that teach leadership, good deeds, and so on.

Keywords: batik pattern, Yogyakarta culture, Ethnomathematics, geometry transformation, local values

Abstrak

Secara umum masih banyak orang yang memandang matematika sebagai ilmu yang jauh dari realitas dan budaya dalam kehidupan sehari-hari. Secara historis, sebenarnya matematika sangat dekat dengan kehidupan sehari-hari dan dikembangkan oleh manusia sebagai jawaban atas fenomena di sekitarnya. Indonesia memiliki budaya yang beragam, termasuk di Yogyakarta. Budaya ini dapat dimanfaatkan untuk mengeksplorasi sejumlah konsep matematika sebagai upaya transformasional untuk mendekatkan matematika dengan realitas dan persepsi masyarakatnya. Selain itu, kita dapat menggunakan budaya sebagai konteks pembelajaran matematika di sekolah. Oleh karena itu, penelitian ini berupaya menggali konsep matematis dari topik transformasi geometri pada motif batik Yogyakarta. Penelitian ini merupakan studi etnografi. Data penelitian dikumpulkan melalui observasi, studi pustaka, dan wawancara dengan praktisi dan seniman budaya batik untuk memahami teknik dan nilai moral, sejarah, dan filosofis batik pada setiap motifnya. Hasil penelitian ini menunjukkan bahwa batik Yogyakarta telah menggunakan konsep transformasi geometri dalam pembuatan motif batik khas Yogyakarta. Selain itu setiap motif atau corak juga mengandung kearifan lokal. Nilai-nilai tersebut adalah nilai moral, sejarah, dan filosofis, yang dapat dirasakan, direfleksikan, dan diterapkan dalam kehidupan sehari-hari, seperti nilai-nilai yang mengajarkan kepemimpinan, perbuatan baik, dan lain sebagainya.

Kata kunci: motif batik, budaya Yogyakarta, etnomatematika, transformasi geometri, nilai-nilai lokal

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Regarding mathematics education problems in Indonesia, a transformational effort is needed to bring mathematics closer to the reality and culture of students. In this case, Ethnomathematics, which was conceived by D'Ambrosio (2016) and based on his concerns with the conditions of mathematics education which is mechanistic, far from the reality and culture of students, can be a solution. Furthermore, Alangui

(2010) has documented that Ethnomathematics succeeded in establishing a relationship between mathematics and the reality of a society, where there originally were gaps as a result of rigid and not contextual informal education. Ethnomathematics learns and combines mathematical ideas, ways, with techniques that are practiced and developed by socio-culture or community culture (Barton, 1999; D'Ambrosio, 2007; D'Ambrosio, 2016; Rosa & Orey, 2016). Through the study of Ethnomathematics, mathematical science can also be found again originating from the different cultural roots of society, so that it can connect to and revive students' critical reasoning and dialogues and can foster students' democratic and tolerant characters by embracing cultural differences and seeing them as opportunities for mathematics education (D'Ambrosio, 2016; Zevenbergen, 2001). Therefore, from many points of view, which create various patterns, Ethnomathematics would be an instrument by which our universe could be better understood as we and others see it (Barton, 1996).

Reflecting D'Ambrosio (2016) on the origin of human knowledge, he understands that every culture develops ways, styles, and techniques to do things in response to any search for explanation, understanding, and learning of a phenomenon occurs. Humans in an area with a specific culture will respond to their environment in a way, style, and technique or develop a knowledge system that they find themselves (D'Ambrosio, 2007; D'Ambrosio, 2016; Rosa & Orey, 2016). Thus, efforts to explain and understand knowledge also depend on observation, comparison, classification, evaluation, quantification, measurement, calculation, representation, and inference in each region and culture. Each area has different culturally established ways of observing, comparing, classifying, evaluating, measuring, counting, representing, and concluding (Rosa et al., 2016; Goetzfridt, 2007).

In each different culture, mathematicians need to be aware of these arts, and techniques that are existing to express understanding, to explain and to study facts and phenomena of the natural and social environment in each of these cultures (Ascher & D'Ambrosio, 1994; Barton, 2007; D'Ambrosio, 2016). All the different knowledge systems that result from the discovery of mathematical concepts in this particular culture are called Ethnomathematics.

Indonesia, rich and diverse in culture, has an opportunity to improve the mathematics education system in Indonesia through transformational efforts to bring mathematics closer to students' reality and culture (Zulkardi, 2002; Abdullah, 2017). Many cultures in Indonesia can be explored to gain context for learning mathematics, including culture in the special region of Yogyakarta, known as the city of culture (Risdiyanti & Prahmana, 2018; Prahmana, 2020). In Yogyakarta, many cultural objects and practices are easily found and can be explored, such as temple buildings, palace, mosque, and batik patterns that have unique and distinctive shapes. Also, it has cultural, historical, and philosophical values that are important for shaping students' character, for example self-confidence, sympathy, empathy, respecting others, awareness of social issues and social-spirited, and responsible (Widodo, 2019). Exploration of mathematical concepts embedded in the culture of the people of Yogyakarta has been documented by several researchers (Risdiyanti & Prahmana, 2018; Rohayati, Karno, & Chomariyah, 2017; Huda, 2018; Pratikno, 2018; Bakhrodin, Istiqomah, & Abdullah, 2019; Ramadani, Praska, & Christian, 2019).

Risdiyanti and Prahmana (2018) explored the concepts of operation of numbers and 2D shapes in traditional games in Yogyakarta. Furthermore, Rohayati, Karno, and Chomariyah (2017) investigated the idea of flat building and building space in the Great Mosque of Yogyakarta. Finally, researchers studied the concept of 2D shapes, 3D shapes, and mathematical modeling found in snacks in Yogyakarta (Huda, 2018), the idea of the assemblage of the Yogyakarta royal palace soldiers (Pratikno, 2018), the concept of the matrix in the Mataram Kotagedhe Mosque building (Bakhrobin, Istiqomah & Abdullah, 2019), and the idea of geometry in the *Sekaten* ceremony in Yogyakarta (Ramadani, Paska, & Christian, 2019).

On the other hand, few researchers still explore mathematical concepts in Yogyakarta's unique and rare batik patterns. This is because these batiks' motif used in this research are only owned by people who have a high position, or their motives can only be found in museums and are rarely found in society. Whereas in the culture of people of Yogyakarta, batik is one of the patterned fabrics that must be worn as a school and civil servants' uniforms on certain days. It has become a fashion style for the people in Yogyakarta when attending official events; it is also used for casual occasions (Didik, 2017; Dian, 2017). The familiarity of batik among the public and students in Yogyakarta is an opportunity for mathematics educators to explore and use it as the context in mathematics learning. Therefore, this study aims to conduct a comprehensive study of mathematical concepts in Yogyakarta's batik patterns that can be used as starting points in learning mathematics in Yogyakarta. Besides that, each motif or pattern often incorporates local values. These, including spiritual, historical, and metaphysical principles, can be sensed, expressed, and implemented in everyday life, such as principles that teach value in leadership, good deeds, and so on. Lastly, the combination of mathematics learning and character education is one of Indonesia's education system goals, so this research's results support this.

The next section of this article describes the ethnographic research methods used to explore the data related to Yogyakarta's batik patterns that contain mathematical concepts. Explaining the results of the exploration of mathematical concepts in Yogyakarta's batik motifs is described in the discussion section. Furthermore, the findings in this study are compared with previous studies from various regions following similar contexts and procedures. Lastly, a research summary shows that mathematical concepts can be used as starting points in learning mathematics written in the conclusions section in Yogyakarta's batik patterns.

METHOD

This study used an ethnographic method, research that studies, and describes a community's culture (Spradley & McCurdy, 1989). This research method was chosen because it is in line with the objectives of Ethnomathematics, namely studying mathematical ideas, processes, and techniques in the culture from the native point of view (Ascher & D'Ambrosio, 1994; Shirley & Palhares, 2016). Research with ethnographic methods involves several activities about a culture that sees, hears, talks, and acts in different ways and in ways that they find themselves (Spradley & McCurdy, 1989). The data collection was carried out by field studies and interviewing the speakers whose selected purposively, namely Didik, a

Yogyakarta batik museum cultural practitioner, and Mrs. Dian, a batik craftsman in the Galur, Lendah, Kulon Progo batik handicraft business. A literature review on batik supplements the results of these observations and interviews. All data were documented in photos, videos, and field notes, then analyzed using the source triangulation technique, and finally described to explore each finding in this study.

In this study, we use three boundaries of the coverage area which are the basis for determining the research subject; community unity consisting of education that speaks one language or accent in the same language, community unity which is limited by the boundaries of an administrative political area and community unity which experiences one experience. the same history, as the limits that have been set in ethnographic research to show the authenticity of culture under study, there is no mixture with other cultures (Flynn, 2008). Therefore, we determined that the community unit to be studied was the original Yogyakarta people who used the same accent, that are the Javanese language, limited by the same administrative area, namely the Special Region of Yogyakarta and experienced the same historical experience, that is history when living, growing and developing in Yogyakarta.

Koentjaraningrat (2015) explains that in ethnographic research there were seven main descriptions produced by ethnographers, that is language, technology systems, economic systems, social organizations, knowledge systems, arts, and religion. In this study, the researcher will focus on one main description, that is the knowledge system because, to dissect the techniques used by the community in making batik motifs, the researcher must observe and dive into the knowledge and art systems to find the knowledge base used in the process of making batik motifs and cultural values contained in the batik motif art. Even so, it does not rule out that other cultural elements will also be studied because they are related to one another.

In conducting ethnographical exploration, researchers begin with four general questions that are the essence of ethnographic principles, that is “where to start looking?”, “how to look?”, “how to recognize that you have found something significant?”, “how to understand what it is?”. Based on these four general questions, the research stages are organized in Table 1.

Table 1. Design of Ethnography Research

General Questions	Initial Answers	Starting Point	Specific Activity
Where to start looking?	In the activities of making batik motif carried out by the people of Yogyakarta where there are mathematical practices in it.	Culture	Conducting interviews with people who have knowledge of Javanese culture in the Yogyakarta community or those who make batik motifs in Yogyakarta.
How to look?	Investigating the QRS (Quantitative, Relational,	Alternative thinking and	Determine what QRS ideas are contained in

	Spatial) aspects of the making batik motif activities of the people of Yogyakarta related to mathematics practice.	knowledge system	making batik motifs activities of the people of Yogyakarta related to mathematics practice.
What it is?	Evidence (Results of alternative thinking in the previous process)	Philosophy of mathematics	Identifying QRS characteristics in the activity of making batik motifs in Yogyakarta society related to mathematics practice. It shows that the activity of making batik motifs for the people of Yogyakarta does have a mathematical character seen from the elements of knowledge and art systems used in everyday life.
What it means?	Valued important for culture and important value patterns for mathematics	Anthropologist	Describe the relationship between the two systems of mathematical knowledge and culture. Describe mathematical conceptions that exist in the activity of making batik motifs for the people of Yogyakarta.

Data collection was carried out by field studies and interviews with informant; Mrs. Dian, the owner of a handicraft batik business in Galur Village, Lendah District, Kulon Progo Regency to see, explore and clarify comprehensively how to make batik motifs along with the basis of knowledge and art used in making batik motifs. Also, we conducted interviews with Mr. Didik, the cultural practitioner at the Yogyakarta Batik Museum, to explore and clarify the cultural values that exist in Yogyakarta batik motifs. The data collection results were analyzed using the source triangulation technique to comprehensively explore the relationship between the mathematical knowledge system and the batik motif culture and to

see the mathematical conceptions that exist in the Yogyakarta batik motif activity. Lastly, the findings are described in the results of this study.

RESULTS AND DISCUSSION

The results showed that in the culture of people of Yogyakarta, there is a patterned cloth or known as batik, which is used in certain traditional events such as pregnancy, marriage, death ceremonies, and so on in the culture of the people of Yogyakarta. In addition, batik has also become a fashion style among people. The Yogyakarta people often use batik when attending official events, or even now, batik is commonly used for casual occasions (Didik, 2017; Dian, 2017). The Yogyakarta provincial government sets batik clothes as mandatory uniforms on certain days in state schools, so students are very familiar with Yogyakarta's typical batik. Currently, every district in Yogyakarta has developed batik with motifs based on each region (Dian, 2017). However, despite the many new modern batik motifs, the Yogyakarta batik pattern or the Ngayogyakarta palace batik pattern, which is the legacy of the Mataram Kingdom's ancestors, is still preserved today. In ancient times, the ancestors made batik patterns carefully. Each style has a visualization of moral values and ideals to be conveyed to the wearer or the general public who see it. Besides, it turns out that in the making of batik patterns they used batik techniques including concepts of geometrical transformation such as mirroring (reflection), shift (translation), rotation, and resizing (dilation), which they learned by self-teaching, based on their experience, creativity, and also cultural traditions. Furthermore, a more detailed explanation of each pattern is as follows.

Values and geometry transformation of Babon Angrem pattern

Babon Angrem or incubating hen is a batik that is usually used by the Yogyakarta people during the seven-month ceremonial for pregnant women. This batik motif symbolizes the love and patience of a mother to her child, who hope the trait can be inherited or imitated later. So, the value that can be taken from the *Babon Angrem* batik pattern is that when you become a parent, you have to give love to the child and be patient in educating the child during his/her growth and development (Didik, 2017; Efianingrum, 2011; Kusrianto, 2013). Besides, in making *Babon Angrem* batik, it used the concept of geometrical transformation in the form of a vertical reflection as shown in Figure 1.



Figure 1. Geometry transformation of *Babon Angrem* pattern (Risdiyanti & Prahmana, 2020)

Values and geometry transformation of Parang Barong pattern

Parang Barong is a batik pattern that was usually used by the aristocrats or used by the king for religious rituals and meditations. However, in its development, this batik can also be used by anyone. The motif in *Parang Barong* batik depicts weapons and power as the power possessed by a knight. The value that can be taken from this *Parang Barong* motif is when one become a knight or leader, he will have the weapon and power that will be his strength, and it should be used for the good and wealth of the people (Didik, 2017; Efianingrum, 2011; Kusrianto, 2013). Also, in making *Parang Barong* pattern, it used the concept of geometrical transformation in the form of translation as in Figure 2.

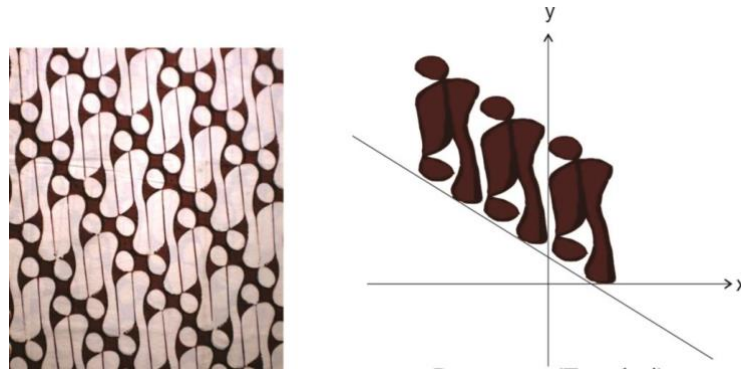


Figure 2. Geometry transformation of *Parang Barong* pattern (Risdiyanti & Prahmana, 2020)

Values and geometry transformation of Parang Klitik pattern

Parang klitik is a batik pattern usually only worn by royal Princesses, but in its development *parang klitik* batik can be used by anyone. This batik motif illustrates the behavior of a princess who is delicate and wise. Thus, the value that can be taken from the motive of *parang klitik* that is a daughter or woman should always have a gentle and knowledgeable temperament in attitude (Didik, 2017; Efianingrum, 2011; Kusrianto, 2013). Besides, in making *parang klitik* pattern, it used the concept of geometric transformation in the form of reflection and translation as in Figure 3.

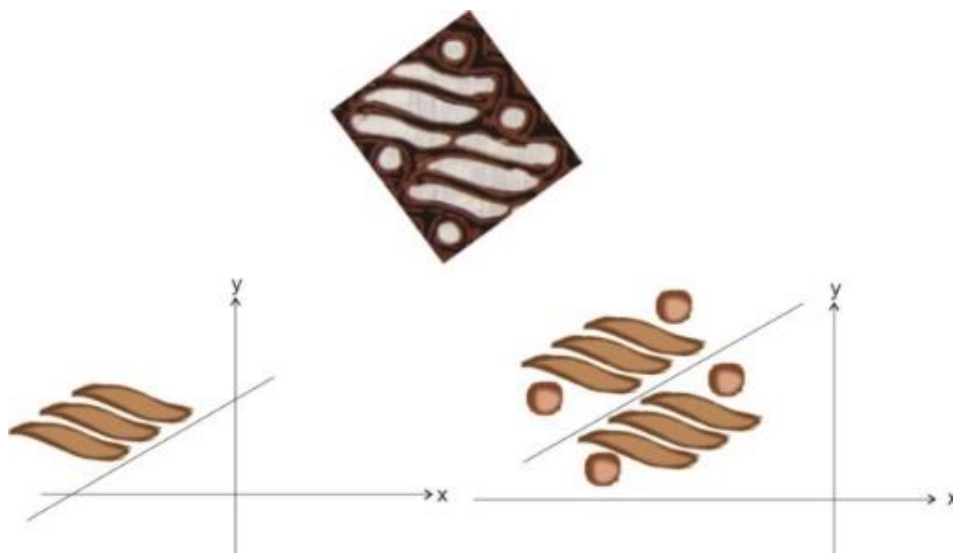


Figure 3. Geometry transformation of *Parang Klitik* pattern (Risdiyanti & Prahmana, 2020)

Values and geometry transformation of *Sidomukti* pattern

Sidomukti is a batik pattern that symbolizes the fulfillment of hope and a prayer. *Sidomukti* comes from the word "sido" which means it really happened or fulfilled the desire and comes from the word "mukti" means happiness and wealth. The value that can be taken from this *Sidomukti* batik is that if you want to get satisfaction and wealth, you should always pray to God and expect His blessing only (Didik, 2017; Efianingrum, 2011, Kusrianto, 2013). In addition, in making batik motifs *Sidomukti* pattern, it used the concept of geometrical transformation in the form of reflection and translation as in Figure 4.

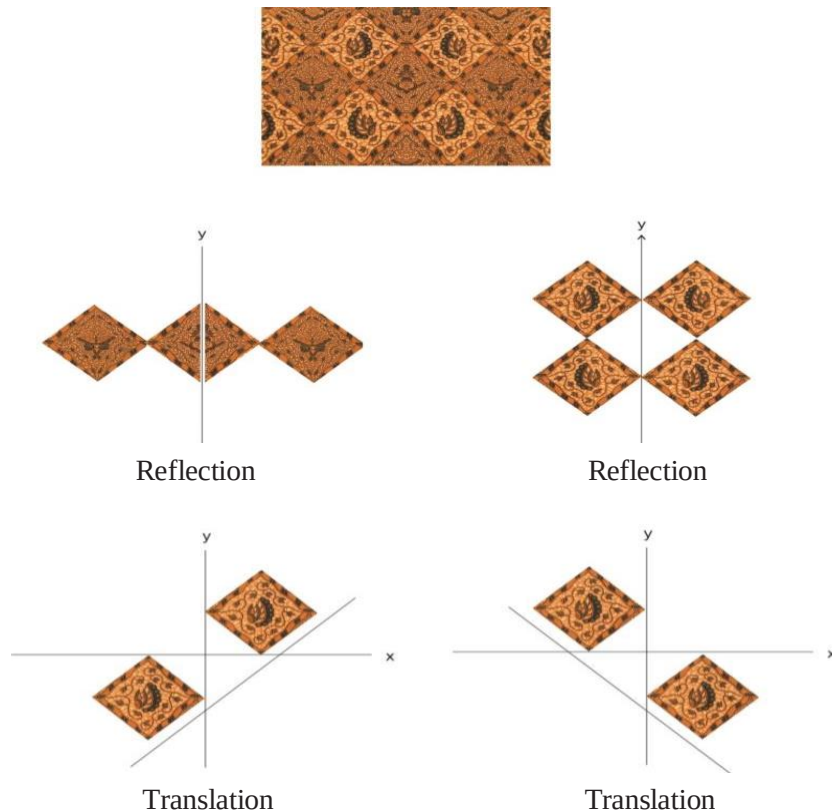


Figure 4. Geometry transformation of *Sidomukti* pattern (Risdiyanti & Prahmana, 2020)

Values and geometry transformation of *Semen Bondhat* pattern

Semen Bodhat is a batik pattern that was initially only used by royal servants. However, in its current development, *Semen Bondhat* batik can also be used by anyone. The *Semen Bodhat* batik motif illustrates the inner and outer peace in life. The value that can be taken from the *Semen Bodhat* motif is that in living a life, you should always look for physical and spiritual friendship (Didik, 2017; Efianingrum, 2011; Kusrianto, 2013). In addition, in making the Batik motif of *Semen Bondhat* batik, techniques have been used by using the concept of geometric transformation in the form of reflection and rotation, as shown in Figure 5.

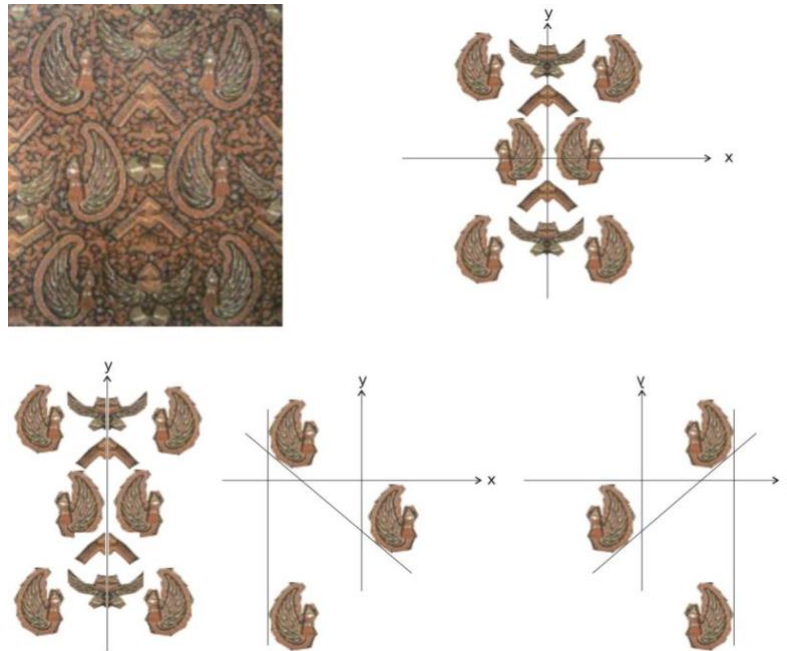


Figure 5. Geometry transformation of *Semen Bodhat* pattern (Risdiyanti & Prahmana, 2020)

Values and geometry transformation of *Sidoluhur* pattern

Sidoluhur is a batik pattern illustrating the qualities of a leader. In *Sidoluhur* batik, several ornaments become the main motives, including building ornaments that adorn the leader's expectations to be in a high and noble position with dignity and respect. Garuda ornaments describe the nature of the courage of a leader inherited from *Suryabrata* (sun). Bird ornament that is depicting a noble character of a low-profile leader. This trait is inherited from the nature of the *Bayubrata* (wind). Flower ornament describes the beauty and happiness gained from a firm, stable foundation, and strong life determination. The ship's decoration is depicting a broadminded leader and calmness in leading and in living a life. Butterfly ornament represents a fair and humanist (Didik, 2017; Efianingrum, 2011; Kusrianto, 2013). The value that can be taken is when becoming a leader, you should emulate the qualities implied in the meaning of each ornament in this *Sidoluhur* batik motif to be able to become a trustworthy leader and be able to prosper the people. In addition, in making *Sidoluhur* batik, it used the concept of geometrical transformation in the form of reflection and rotation as shown in Figure 6.



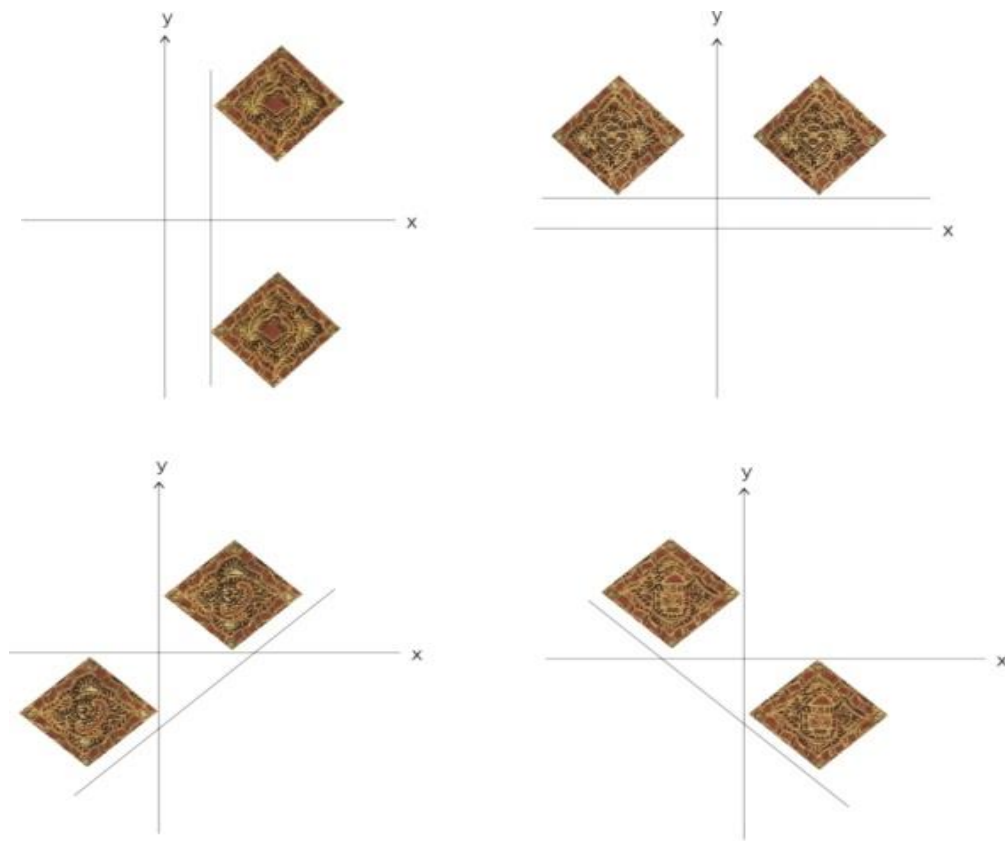
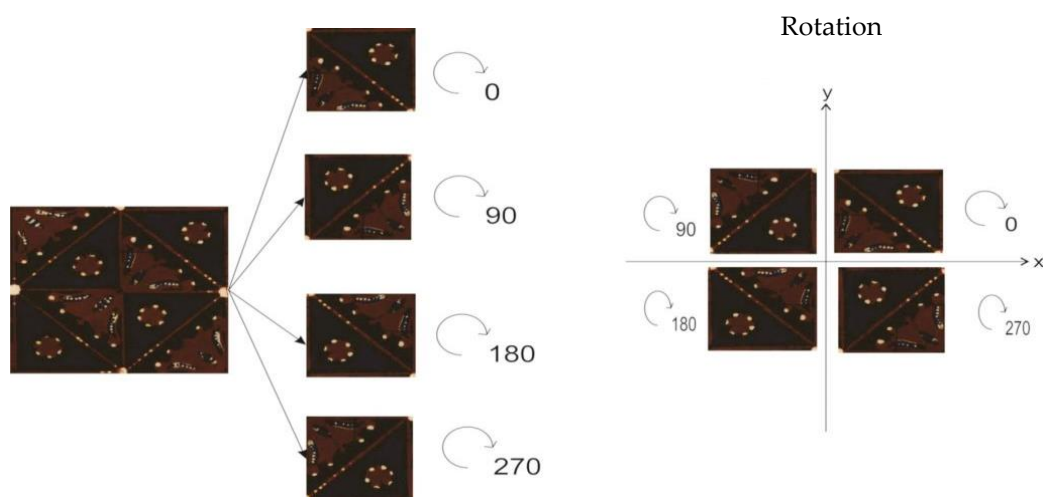


Figure 6. Geometry transformation of *Sidoluhur* pattern (Risdiyanti & Prahmana, 2020)

Values and geometry transformation of Soblog pattern

Soblog is a batik pattern that illustrates determination, accuracy, and patience. Thus, the value that can be taken in the *Soblog* batik motif is that if you have a desire or dreams, then try to achieve it with full determination, accuracy, and patience (Didik, 2017; Efianingrum, 2011; Kusrianto, 2013). In making batik motifs *Soblog* batik, it used the concept of geometric transformation in the form of reflection and rotation, as shown in Figure 7.



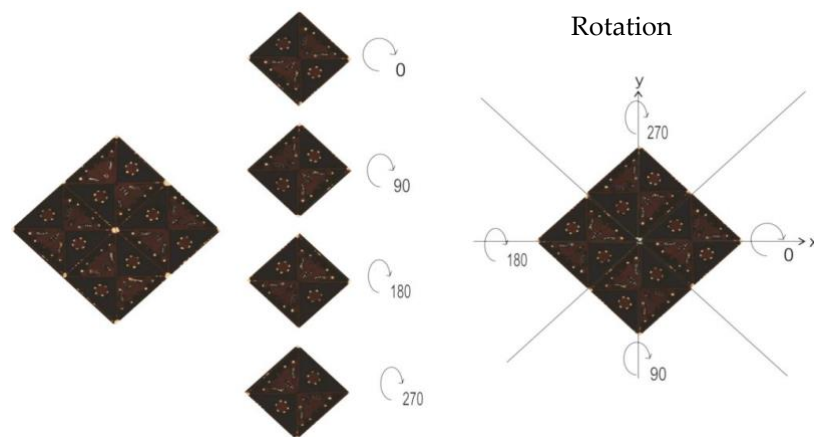


Figure 7. Geometry transformation of *Soblog* pattern (Risdiyanti & Prahmana, 2020)

Values and geometry transformation of *Sidowirasat* pattern

Sidowirasat is a batik pattern that was usually used by parents in their children's marriages. The *Sidowirasat* batik motif illustrates that parents always give advice and guide the bride and groom to start a marriage. Thus, the value that can be taken from this *Sidowirasat* motive is that when a child has chosen to marry, parents have genuinely let go of their children, but parents are still there to guide and give advice in their marriage (Didik, 2017; Efianingrum, 2011; Kusrianto, 2013). In addition, in making batik motifs *Sidowirasat* pattern, it used the concept of geometric transformation in the form of reflection and translation as in Figure 8.

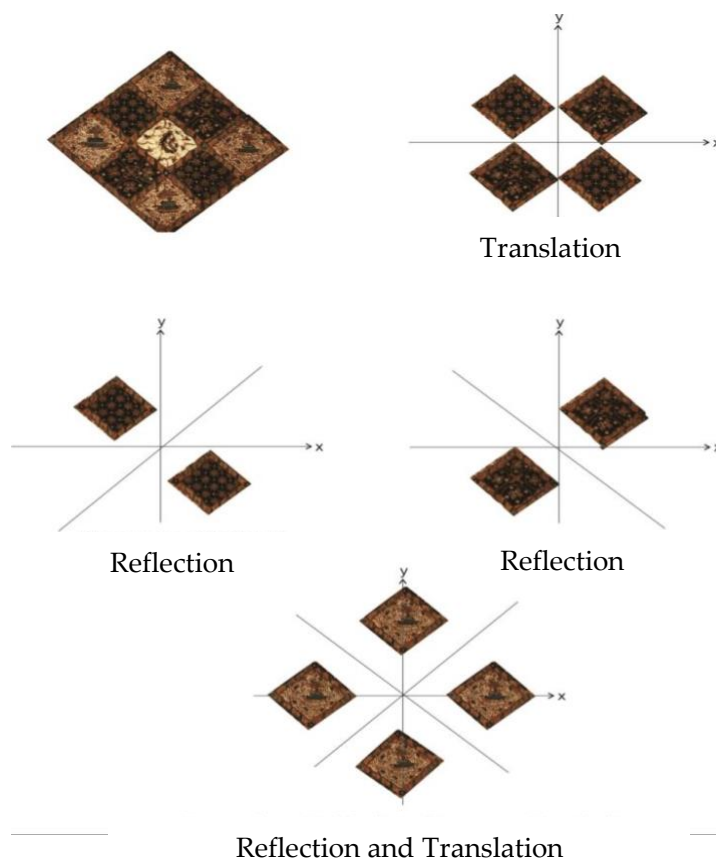


Figure 8. Geometry transformation of *Sidowirasat* pattern (Risdiyanti & Prahmana, 2020)

The results of the exploration of mathematical concepts in making Yogyakarta batik motifs show that Yogyakarta society has used the concept of geometric transformation that they learned by themselves and these creative ideas emerged from their experiences in determining the forms of Yogyakarta batik motif (Didik, 2017; Dian, 2017). Based on the history, before Indonesia's independence, Indonesia consisted of kingdoms including in Yogyakarta there was the Mataram kingdom (Didik, 2017). The culture of the Mataram kingdom community believes in an ideology, values, norms, manners, and ethics that are inherent in the daily life of the community and they have a desire that what they believe is also believed by future generations so that social order will not go far of the social order during the kingdom. Therefore, they try to pass down their ideology, their beliefs through any media including through Yogyakarta's unique batik motifs (Didik, 2017). The results of the research showed that in every Yogyakarta batik motif, it contained moral messages containing ideology, values, norms, ethics and ethics that govern how to relate to humans, how to relate to nature, how to lead and others in living life (Didik, 2017).

Indonesia has extracted Ethnomathematics from various places and cultural activities in Indonesia, such as the concept of sets and number operations in the process of determining the sacred day in Sambeng village, Cirebon (Abdullah, Maulidia, & Amelia, 2019), the concept of geometric transformation in Balinese woven handicrafts (Puspawati & Putra, 2014), the idea of a 2D shapes in the Jamik Mosque building in Bengkulu city (Lusiana, Afriani, Ardy, & Widada, 2019), the symmetry concept and circles on the Marawis musical instrument (Marina & Izzati, 2019), the concept of comparative value in the activities of cocoa farmers in Temuasri Sempu Banyuwangi Village (Aprilianti, Sunardi, & Yudianto, 2019), and the concept of 2D shapes on Tapis fabric and Lampung Traditional House (Loviana, Merliza, Damayanti, Mahfud, & Islamuddin, 2020). The findings of Ethnomathematics exploration have also been applied in mathematics learning in several schools in Indonesia. It has been proven that mathematics learning based on mathematics can change students' views about the relationship between mathematics and reality and their culture. When students understand such a relationship, students experience lower math anxiety.

Until today, we see that many Indonesian students experiencing mathematics anxiety. We see that this experience is related to not associating mathematics in reality and culture. Among the feelings they experienced are feeling of scared, that mathematics is distant to their life, a feeling of meaningless in life, and unable to understand the benefit of learning mathematics to solve problems in daily life (Karnilah, Juandi, & Turmudi, 2013; Hendriana, Prahmana, & Hidayat, 2019).

The results of this study and previous research on the exploration of Ethnomathematics in Indonesian culture can be one of the transformative efforts to bridge mathematics with the reality and perception of students in learning mathematics. Thus, students can see that mathematics is close to the culture around them, and students can finally take on the meaning of the mathematical concepts they learn to apply in their daily lives (Abdullah, 2017; Risdiyanti & Prahmana, 2018; 2020). To be more attuned to

life in modern democratic communities, if there are aspirations to change mathematics instruction, then this aspect of mathematics education needs to be better understood to be better taught (Bishop, 2007).

Changes in students' views about the relationship of mathematics with reality and culture around them after learning mathematics using the context of the exploration of Ethnomathematics were shown from previous researchers. They have succeeded in designing mathematics learning by using the context of the exploration of Ethnomathematics. They have proven to increase students' understanding and make students feel that the mathematics they learn is meaningful. These results are similar to findings of several previous research (Nasrullah & Zulkardi, 2011; Wijaya, Doorman, & Keijzer, 2011; Risdiyanti, Prahmana, & Shahril, 2019). Nasrullah and Zulkardi (2011) explored learning number operations using “*bermain satu rumah*” game, a traditional game from South Sulawesi, Indonesia. Furthermore, Wijaya, Doorman, and Keijzer (2011) are intended to contribute to developing and formulating a local instruction theory on linear measurement in second grade of primary school in Indonesia through Indonesian traditional games namely playing *gundu* and *benthic*. Lastly, Risdiyanti, Prahmana, and Shahril (2019) produced social arithmetic learning designs using the Kubuk Manuk game.

Ethnomathematics exploration is not only carried out by researchers and educators in Indonesia, but also by researchers in many countries. The examples are exploring the concept of numbering in traditional game of Vhaveda tribe, Afrika Selatan (Nyoni, 2014), mathematical modeling in the Brazil's largest carnival (de Freitas Madruga & Biembengut, 2015), mathematical modeling in braids craft of Argentina society (Albanese, 2015), number operations, symmetries, mathematics symbol and measurement concept in some traditional game of Boudin Arab Tribe such as Ta'ab Stick game, Mozkat-5 stone game and Sega game (Fouze & Amit, 2017), concept of circle and transformation of the plate motifs used in the culture of quill acing as tribe in Southern Colombia, Amerika (Araújo, 2015). Atweh et al (2007) conducted a study on Ethnomathematics in Community Bank to determine pedagogical interventions that must be done to overcome difficulties in credit handling and product control management in Community Bank. This shows that researchers in many countries in the world have realized the importance of restoring mathematics to its origins and exploring Ethnomathematics in the culture of society in each country. They do not only explore but also conduct studies on the curriculum in their respective countries and try to see the opportunities and effectiveness of including Ethnomathematics in the educational curriculum.

Several studies on Ethnomathematics in the educational curriculum, such as research by Gonçalves & Pires (2014) which studies the curriculum in professional high school in Brazil and possibilities to including Ethnomathematics inside it. Pinto (2017) reviewed the draft Base Nacional Commun Curricular (BNCC) in Brazil which is the curriculum accompanied by traditions that was always updated in accordance with the social condition in which their traditions emerged. Ethnomathematics were included in this curriculum. Osorio (2016) discuss the importance of the indigenous Gunadule school curriculum min alto Cayman, South America considering the sociocultural, Ethnomathematics, and inculturation perspective. Dos Santos and da Silva (2016) showed that at the Quilomba School in Brazil there is a need

to provide access to mathematical knowledge with Ethnomathematics collaboration. In Indonesia, Ethnomathematics is just being explored. Researchers are trying to make a mathematics learning design based on Ethnomathematics, but has not yet reached the stage of trying to introduce it to the government through the education curriculum. We believe that this needs to be done to see the success of the Ethnomathematics-based application of mathematics learning in Indonesia. We believe that it can overcome some of the problems that students encounter in learning mathematics such as math anxiety, misconceptions, misunderstandings, disinterest in mathematics and others.

There is no question that mathematics is and must be practiced as an essential part of human understanding. There is also no question that its position is currently a restricted version of its potential in many countries and cultures. Education in mathematics is intended to encourage mathematics comprehension for everyone. The status and roles of mathematics in our society need to be modified to emerge in this situation. An ethnomathematical understanding of mathematics education helps this transition (Barton, 1996).

CONCLUSION

Ethnomathematics in Indonesia does not only stop at cultural exploration and experiments on mathematics learning in several schools but also in the future, we can introduce it to mathematics education curriculum in Indonesia. The people of Yogyakarta have used the concept of geometry transformation in making batik patterns such as in the Babon Angrem, Parang Barong, Parang Klitik, Sidomukti, Semen Bondhat, Sidoluhur, Soblog, and Sidowirasat motifs. Knowledge of the concept of geometrical transformation has been used and self-studied by the people of the Mataram kingdom in Yogyakarta. Despite mathematical contents, batik has moral, historical, and philosophical values in each style that can be felt, reflected, and applied in daily life, such as values in leadership, good deeds, and so forth. A comprehensive study of the culture of the Yogyakarta community to discover the concept of geometry transformation can be used as a starting point in mathematics teaching and learning activities in Yogyakarta. It can be implemented to improve understanding of geometry transformation for students who live in rural and urban areas.

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CONTEXTUALIZED LEARNING MODULES IN BRIDGING STUDENTS' LEARNING GAPS IN CALCULUS WITH ANALYTIC GEOMETRY THROUGH INDEPENDENT LEARNING

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Abstract

The transition of the educational system in the Philippines vastly affects basic and higher education. A mismatch of pre-requisite Mathematics learning competencies from the basic education level occurred when the student reached higher education. This descriptive-developmental method of the study utilized the developed contextualized learning modules for the bridging course on the identified learning gaps in Calculus with Analytic Geometry for the Bachelor of Secondary Education (BSEd) major in Mathematics. Real-world concepts and situations featuring the Province of Sorsogon, Philippines were integrated into the learning modules while promoting independent learning. The content, format, presentations and organizations, accuracy, and up-to datedness of information of the learning modules passed the evaluation of 13 experts (Mathematics Professors) from the different Higher Education Institutions (HEIs) in the Bicol Region, Philippines. Also, the 18 student participants were very much satisfied with the utilization of the learning modules that bridged their learning gaps in the conic section through independent learning.

Keywords: conic sections, bridging course, learning gaps, independent learning, contextualized learning modules

Abstrak

Transisi sistem pendidikan di Filipina sangat mempengaruhi pendidikan dasar dan tinggi. Ketidaksesuaian kompetensi pembelajaran Matematika prasyarat dari tingkat pendidikan dasar terjadi pada saat siswa mencapai perguruan tinggi. Metode pengembangan deskriptif dari studi ini memanfaatkan modul pembelajaran kontekstual yang dikembangkan untuk menjembatani suatu topik pada kesenjangan pembelajaran yang diidentifikasi dalam Kalkulus dengan topik Geometri Analitik untuk Sarjana Pendidikan Menengah (S.Pd.) jurusan Matematika. Konsep dan situasi dunia nyata yang menampilkan Provinsi Sorsogon, Filipina diintegrasikan ke dalam modul pembelajaran sambil mempromosikan pembelajaran mandiri. Isi, format, presentasi dan organisasi, keakuratan, dan kemutakhiran informasi modul pembelajaran telah melewati proses evaluasi dari 13 ahli (Profesor Matematika) yang berasal dari berbagai Institusi Pendidikan Tinggi (IPT) di Wilayah Bicol, Filipina. Selain itu, 18 peserta didik peserta sangat puas dengan pemanfaatan modul pembelajaran yang menjembatani kesenjangan pembelajaran pada bagian berbentuk kerucut melalui pembelajaran mandiri.

Kata kunci: irisan kerucut, matrikulasi, kesenjangan pembelajaran, pembelajaran mandiri, modul pembelajaran kontekstual

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Mathematics is considered the universal language, but student math achievement is exceptionally low (Morita-Mullaney, Renn, & Chiu, 2020; Wong & Chan, 2019; Maloney, Ramirez, Gunderson, Levine, & Beilock, 2015). The complexity of mathematics and its utility is not as obvious as it is (Andrews & Hatch, 1999) as an abstract discipline with various applications in different areas (Jankvist, 2015). Relating mathematics concept to its context in the real – world is now the demand (Gainsburg, 2008; Jurdak, 2006) which allows learning transfer through contextual teaching and learning approach (Boaler, 1993; Mahendra, 2016; Mulyono & Lestari, 2016; Nabila & Widjajanti, 2020).

The process of matching the content and instructional strategies relevant to students is called contextualization (Department of education [DepEd], 2016). It is the use of relevant and meaningful things, situations, and experiences to students in the presentation and discussion of the learning content. Contextualization can help enable the transformation and construction of a larger motivational environment for students (Haris & Putri, 2011; Weinberg, Besile, & Albright, 2011). When mathematics is real and accessible, students overcome their fear which leads to an appreciation of its utility (Wang, Sun, & Wickersham, 2017). Realistic mathematical approaches increase the learning independence of students (Hasibuan, Saragih, & Amry, 2019).

Context is directly related to everyday life (Supiyati, Hanum, & Jailani, 2019). Mathematizing culture and environment where it is assumed to motivate students to recognize mathematics as part of everyday life aim to enhance the student's ability, deepen their understanding of all forms of mathematics, and to make meaningful mathematical connections (Alangui, 2017; Muhtadi, Sukirwan, Warsito, & Prahmana, 2017; Rosita, 2016). Albeit, contextualization consumes much time for preparation with the unavailability of the local materials and difficulty in pedagogy. Not all topics are applicable and students' differences pose a challenge to contextualization, however, contextualization still increases the learning engagement wherein students have better retention of concepts and conceptual understanding.

Scaffolding is an integral part of contextualization (Howe, 2013). The first step in scaffolding is contextual support followed by a series of actions and interactions to balance the routine of the scaffolding procedure (Amerian & Mehri, 2014). Scaffolding is important to students with learning difficulties because it increases their mathematical understanding (Sutiarso, Coesamin, & Nurhanurawati, 2017; Gravemeijer, 2011). An increase in students' engagement characterizes high-quality teaching and learning (Ashwin & McVitty, 2015). Instructional support is the most dominant factor determining student engagement (Alrajeh & Shindel, 2020) and learning becomes more efficient and relevant to them (Baker, Hope, & Karandjeff, 2009). Students also develop new skills, knowledge, abilities, and attitudes (Rivet & Krajcik, 2008).

The learning module contains summaries of material, training, and covers how students build knowledge (Hamdunah, Yunita, Zulkardi, & Muhafzan, 2016). It is an instructional material used to ease, encourage, improve, and promote teaching and learning activities to improve and facilitate effective processes of instruction (Matarazzo, Durik, & Delaney, 2010; Fradd, Lee, Sutman, & Saxton, 2001). Learning module offers new approaches and learning opportunities that enhance student's knowledge and helps them overcome deficiencies (Gordon & Nicholas, 2013). Teachers should be aware of the prerequisite topics so that intervention could be done to address students' least mastered competencies (Herrera & Dio, 2016). Learning modules develop mathematical skills (Setyani, Putri, Ferdianto, & Fauji, 2020), and improve students' abilities and affect their learning motivation (Saifiyah, Ferdianto, & Setiyani, 2017). Nevertheless, contextualized learning modules are effective to bridge learning gaps independently as they supplement and complement the teacher's verbal explanations in

making a learning experience. Contextualized learning modules can also clarify, vitalize, emphasize the instruction, and enhance learning in the process of transmitting knowledge, ideas, skills, and attitude (Oladejo, Olosunde, Ojebisi, & Isola, 2011). The integration of contextualization in the learning modules allows greater transfer of mathematics ideas related to their environment (Boaler, 1993; Anwar, Budayasa, Amin, & Haan, 2012). As students take ownership of their learning, they are engaging in independent learning (Field, Duffy, & Huggins, 2015; Livingston, 2012) which has a large effect on student learning (Chingos & Whitehurst, 2012; Olayinka, 2016). Contextualized learning modules are user – friendly where students can learn on their own (Lai & Hwang, 2016) and are widely accepted by the present educational systems to have a positive effect on learning (Hendriana, Prahmana, & Hidayat, 2019). This connotes that contextualized learning modules eventually promote independent learning.

The full implementation of the Philippine new K to 12 Basic Education program in 2016 is a move to meet the world's demand for quality education at the basic education level. The education transition from 2016 to 2021, presents significant challenges in the reform of the Philippine educational landscape (Commission on Higher education [CHED], 2020) such as student learning gap when they reach the higher education level. Learning gaps occur when there is a mismatch of learning competencies between the basic education and higher education levels. In this light, this paper aimed to developed contextualized learning modules for a bridging course in Calculus with Analytic Geometry that promotes independent learning among the Bachelor of Secondary Education (BSEd) major in Mathematics students who are non-STEM (Science, Technology, Engineering & Mathematics) strand graduates during the Senior High School.

METHOD

This descriptive – developmental study describes a phenomenon and its characteristics (Nassaji, 2015) and assesses the changes with a phenomenon over an extended period (Jaikumar, 2018) brought by the developed contextualized learning module in bridging student's learning gaps. ADDIE model (Analyze, Design, Develop, Implement, Evaluate) is the main approach used in evaluating the characteristics of the developed learning module (Cullata, 2020). The ADDIE approach is a product developmental concept applied for constructing performance-based learning adopting the Input – Process – Output (IPO) paradigm (Branch, 2009).

Analyze phase

The Philippine CMO No. 105, s. 2017 mandated Higher Education Institutions (HEIs) not to discriminate against college enrollees (CHED, 2017). Furthermore, the Philippine CMO No. 75, s. 2017 is the Policies, Standards, and Guidelines (PSG) for Bachelor of Secondary Education (BSEd) including specialization in Mathematics (CHED, 2017). BSEd- Math accommodated STEM (*Science, Technology, Engineering, and Mathematics*) graduate students and Non – STEM (*all other senior high*

track/ strand) (DepEd, 2014). Initially, STEM graduates are expected to enroll in science and mathematics-related courses including BSEd – Math. The learning gap occurred to BSEd – Math students who are non-STEM graduates of Senior High School. Curriculum mapping was conducted to Calculus with Analytic Geometry to all SHS strands, and conic sections were the identified learning gap as a prerequisite to the subject. The identified learning gap was also validated by tertiary mathematics professors teaching Calculus subjects.

Design phase

The learning modules on the identified student learning gaps about conic sections were designed by contextualizing mathematical concepts to reality by bringing Sorsogon's best in the materials. Contextualization is present in various parts of the module that promotes independent learning (Hasibuan, Saragih, & Amry, 2019; Yilmaz, 2020) where content and strategies are matched to relevant and meaningful things, situations, and experiences. Besides, the module uses clear and simple language, presentation, and illustration to cater to all students. It is designed to be used as supplementary material in the Calculus with Analytic Geometry bridging course to second-year Non-STEM graduates BSEd – Mathematics students.

Develop phase

The collated relevant materials which are locally available to conic section concepts were used in the development of the module through its integration in the following parts of the module: introduction, lesson proper, examples, formative and summative assessment. Contextualization of modules featured things familiar to learners such as tourist attractions, products, and situations to the locale of students in the Province of Sorsogon, Philippines. Language experts and design experts were consulted to improve the material's language and medium of delivery as well as the presentations of figures and illustrations. Recommendations of mathematics experts who are teaching the subjects were also incorporated in the development of the contextualized learning modules.

Implement phase

The researcher conducted the bridging course to the purposively selected 18 BSEd – Mathematics major student-participants of the Sorsogon State College at the beginning of the second semester, SY 2019-2020. These student-participants are the non-STEM graduate of Senior High School who was subjected to the bridging class. The bridging class was conducted for 18 hours which were distributed evenly for 3 weeks. The general orientation of the purpose, mechanisms, and schedules was conducted first during the first hour of the class before the distribution of the learning modules. Students were tasked to solely read and understand the content of modules, answer the assessments, and explain their solutions and answers. The learning modules were implemented in series from module 1 to module 4 which cover circle, parabola, ellipse, and hyperbola, respectively.

Evaluate phase

This phase assesses the quality of instructional products and processes (Branch, 2009; McGriff, 2000; Kurt, 2018) which is continuous in conjunction with the first four phases of the ADDIE model (Branch, 2009). The developed learning modules were sent to 13 experts (Mathematics Professors – teaching Calculus) from seven HEIs in Bicol Region (Region V) who served as evaluators using the adopted Evaluation Rating Sheet for Print Resources from Guidelines and Process for LRMDs (*Learning Resources Management and Development System*) Assessment and Evaluation (DepEd, 2009).

Jurors evaluated the modules using 5 – point Likert scale for content, format, and presentation, and organization, and 4 – point Likert scale for accuracy and up – to – datedness of information (DepEd, 2009). Experts' validation made modules meet the valid criteria (Hasibuan, Saragih, & Amry, 2019). Indicator's accumulated points were interpreted based on set standards of the instrument – at least 21 of 28 (contents); 54 of 72 (format); 15 of 20 (presentation and organization); and 18 of 24 (accuracy and up – to – datedness of information) (DepEd, 2018). Contextualization feature indicators were adopted from its definition stipulated in DepEd Order No. 35 (2016), and independent learning feature indicators were also adopted from Mckendry and Boyd (2012) in the expert evaluation.

After conducting the bridging course, the students evaluated the acceptability of the learning modules using the instrument of Tan-Espinar and Ballado (2017) with a 4 – point Likert scale. The students' comments were also analyzed to support their written evaluation. On the other hand, students' evaluation measured the acceptability of clarity, usefulness, suitability, adequacy, timeliness, and language, style, format, illustration, and presentation. Frequency count and weighted mean were used in the analysis and quantification of the data through Microsoft Excel and interpreted using the interval used from the study of Owusu-Manu, Torku, Pärn, Addy, and Edwards (2017). All numerical results are supported by the content analysis of the comments and suggestions of the experts and students who utilized the learning module.

RESULTS AND DISCUSSIONS

Learning Gap of Non – STEM students to Calculus with Analytic Geometry

Philippines CMO No. 75, s. 2017, the adapted course description of Calculus with Analytic Geometry is: *The course equips the students with knowledge and skills needed to be able to determine limits of the functions, to differentiate and integrate algebraic, exponential, logarithmic, and trigonometric functions in one variable. It also includes exposure to more challenging problems covering the continuity and areas of the region.* Explicitly, it includes more challenging problems about areas of the region. This topic is one of the major practical applications of the integration of calculus itself. College Algebra, Trigonometry, and Geometry are prerequisite subjects of Calculus with Analytic Geometry, but conic sections (*circle, parabola, ellipse, and hyperbola*) are not included in any

of these subjects and even in Calculus with Analytic Geometry. The course description of Calculus with Analytic Geometry focuses more on limits and continuity, differential, and integral calculus.

Conic sections are topics in Pre – Calculus subject in SHS - STEM, other SHS strands/ tracks do not offer this subject (Ascano, Martin, Olofernes, and Tolentino, 2016). This entails that this topic is a learning gap for all non – STEM graduates studying BSEd – Mathematics. Book authors proved that conic sections are in the topic area of the region bounded by the curve. Anton, Bivens, and Davis (2012) used the concepts of conic sections in their published book to determine the area of the region bounded by equations. *Find the area of the region that is enclosed between the curves $y = x^2$ and $y = x + 6$ (Example 2, p.415). Find the area of the region enclosed by $x = y^2$ and $x = y + 2$ (Example 4, p. 417). Show that the area of the ellipse in the accompanying figure is πab (Exercise Set 6.1, no. 50, p. 420).* These examples show that the concepts of conic sections are prerequisites for the area of the region bounded by equations/curves. The first two examples are parabolas that open upward($y = x^2$) and rightward($x = y^2$). The latter example is to prove the formula of ellipse ($\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$) using integral calculus.

Larson and Edwards (2019) also used conic sections to illustrate the concept of areas of the region bounded by equations/curves in their Calculus book. *Find the area of the region bounded by the graphs of $f(x) = 2 - x^2$ and $g(x) = x$ (Example 2, p. 446). Find the area of the region between the graphs of $f(x) = 3x^3 - x^2 - 10x$ and $g(x) = -x^2 + 2x$ (Example 4, p. 447). Find the area of the region bounded by the graphs of $x = 3 - y^2$ and $x = y + 1$ (Example 5, p.448).* These examples used parabolas to illustrate areas of the region bounded by equations/curves. The first function in Example 2 is a parabola that opens downward, and so with the second equation in Example 4. Hence, the first equation in Example 5 is a parabola that opens leftward.

Furthermore, Stewart (2016) also used conic sections to illustrate the area of regions. More of his examples are also used in parabolas which require basic integration. In Chapter 7 of Stewart's book (Techniques of Integration) Exercises 7.3 (pp. 491 – 492, no. 34), one of the exercises stated *Find the area of the region bounded by the hyperbola $9x^2 - 4y^2 = 36$ and $x = 3$.* He also added that in determining the area of circle and ellipse, an integral of the form $\int \sqrt{a^2 - x^2} dx$ where $a > 0$ will be used. This requires a technique of integration, specifically trigonometric substitution.

Contextualized Learning Modules for Bridging Course in Calculus with Analytic Geometry

In various parts of the modules, different things relevant to the students were used to illustrate and transfer the concepts of conic sections and eventually evaluate their context. The use of local material's contexts in the introduction is a platform to ensure the learning transfer of everyday mathematics to abstract mathematics (Boaler, 1993).

The circle is traced in the image of coconut (*locally called "niyog" in the entire Philippines, and "lubi" in the local dialect of Sorsogon*"), which is put in the Cartesian plane labeled the parts and is

used to introduced circle conic sections in the module. From real objects, the students learned the abstract concepts of the circle, then eventually apply back to situational problems. One of these real and contextualized objects is “*timitim*” (see [Figure 1](#)) which is one of the local delicacies of Gubat, Sorsogon (Villareal, 2018; Sorsogon 101, 2019).



Figure 1. Sorsogon “*Timitim*”

The word “*timitim*” is derived “*patikim – tikim*” or to taste. Its main ingredient is “*kamoteng kahoy*” or cassava. Since it is locally made in Sorsogon, students easily recognized it and grasp the context of the circle based on the given scenario. Contextualization is also integrated into assessments, like the circles on the ceiling of Sts. Peter and Paul Cathedral in Sorsogon City, Sorsogon. Contextualization is an avenue for students to appreciate, and an authentic transfer of mathematics concepts.



Figure 2. Sorsogon “*Parola*”

Parabola is presented by its trace in the image of Fatima Church in Sorsogon City, Sorsogon. Similarly, after abstraction, students applied it back into realistic situations, such as the arch (somewhat parabolic) of the doorway of Lighthouse or “*Parola*” (see Figure 2) at Sorsogon City Boulevard or locally called “Rompeolas”. Parabola concepts are also applied to other different situational problems such as radio signal satellite dish, the cable of suspension bridges, sound reflector, bridge arch, and television dish antenna.

Meanwhile, an ellipse is introduced using the orbit of the moon around the earth, orbits of planets around the sun, the elliptical image of the earth, and castle nut scientifically called “*canarium ovatum*” and locally called “*pili*” (*entire Philippines*) (Yve, 2015). Pili (see Figure 3) uses its trace to introduce the concept of the ellipse, and its determined parts are used further in abstraction. Furthermore, the concept of an ellipse is also applied to pili sculpture (at Sorsogon City Boulevard) and its platform, situational planet’s orbit, semi-elliptical dome and tunnel, Colosseum at Rome, and Sputnik I’s (artificial satellite) orbit.



Figure 3. “*Pili*” Fruit

Different sceneries such as an image in the middle of Bulusan Lake at the foot of Mt. Bulusan, Bulusan, Sorsogon, the mountains around the lake, and its reflection to the water traces the image of the hyperbola and the hyperbola image at the ceiling of Sts. Peter and Paul Cathedral were used as a platform for the abstraction of the hyperbola concept and used also in scenarios for real-world application of the concept. Added to these scenarios are a hyperbolic mirror for panoramic photography, explosion scenario, hyperbolic shaped pillar, long-range navigation (LORAN), and the aerial view of crossroads at Sorsogon City (see Figure 4).

The mind naturally seeks meaning in a context related to a person's current environment that makes sense and is useful (Ampa, Basri, & Andriani, 2013). The contextualization feature relates to the environment in the transfer of mathematics ideas (conic section) to students. Beginning at context is

meaningful and fun, it also sharpens students' minds and focuses (Muslimin, Putri, Zulkardi, & Aisyah, 2020). Activities are put into perspective and linked concepts into practical real-world contexts (Dewi & Primayana, 2019; Lasa, Abaurrea, & Iribas, 2020). In this way, content becomes meaningful and relevant to them (Baker, Hope, & Karandjeff, 2009). Through this, learning engagement and conceptual understanding increase, and students have better retention of concepts (Reyes, Insorio, Ingreso, & Hilario, 2019).

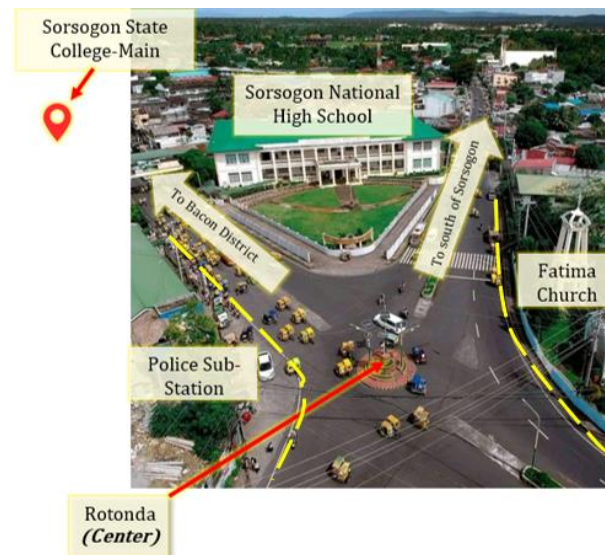


Figure 4. Sorsogon City Crossroads

Realistic mathematical approaches increase the learning independence of students (Hasibuan, Saragih, & Amry, 2019). Realistic mathematics' principle is the transition from informal knowledge to formal knowledge through contextual problems (Yilmaz, 2020). Also, Independent learning or independent study is a process, method, and philosophy of education in which a student acquires knowledge by his or her efforts, and develops the ability for inquiry and critical evaluation (Field, Duffy, & Huggins, 2015). Here, learners have ownership and control of their learning (Livingston, 2012). Using the modules, students independently learn the lessons.

The modules use precise and simple language (Jamison, 2000), concrete illustrations, the relation of each mathematics concepts to explain the context of conic sections. Moreover, color coding is also used as a cueing instrument to highlight which gives ease to readers in the algorithm, such as in similar terms, and related mathematical concepts (Maile & Cooper, 2018). Furthermore, related ideas/ concepts/ mathematical terms are summarized into table/s which enable the readers to see and grasp the relatedness of each concept to one another. The relatedness of information gives users a general outlook on how to execute the step by step solutions. Solutions of examples are readily prepared with complete and comprehensive explanations to guide the readers. Fully packaged learning solutions can be for independent learning where students work step – by – step (Utemov, Khusainova, Sergeeva, & Shestak,

2018). Final answers to problems of formative assessments are readily available at the back so that students can self-assess and monitor their progress (Biehler, Fischer, Hochmuth, & Wassong, 2010). These make the modules to be student-friendly wherein students learn on their own (Dewi & Primayana, 2019). The integration of independent learning guides the students to address the gap and deficiencies independently (Biehler, Fischer, Hochmuth, & Wassong, 2010). Students discover facts by themselves and become responsible for their learning (Oginni & Owolabi, 2012).

Jurors' Evaluation of the Learning Modules

Evaluation is one of the major determinants of the quality of developed learning modules. It also determines the value of worthiness, appropriateness, goodness, validity, and legality of modules (Kizlik, 2014). The summary of jurors' evaluation of the learning modules is presented in Table 1. All learning modules accumulated passing points from each criterion.

Table 1. Jurors' summary of points of learning modules evaluations

Criteria	Points to pass (DepEd, 2009)	Mean Scores of Jurors' Evaluation			
		M1	M2	M3	M4
Content	at least 21 of 28	26	25	24	24
Format	at least 54 of 72	66	63	62	62
Presentation and organization	at least 15 of 20	18	17	18	17
Accuracy and up-to-datedness of information	at least 18 of 24	22	22	22	22

Legend: M1, M2, M3, M4 – Modules developed

This infers that content is suitable for students' level of development. The scope, range, and depth of content and topics are appropriate to their learning needs. The level of difficulty of modules is appropriate for their age and stage of learning, and the level of detail is appropriate for the achievement of the specified learning outcomes. The result tells that the learning modules contributed and supported the achievement of the specific objectives (Rosita, 2016). It also reinforces, enriches, and/or leads mastery of competencies. The module develops higher cognitive skills of students such as critical thinking, creativity, inquiry, problem solving, and others (Dewi & Primayana, 2019).

As to the format of the modules, findings show that the modules' prints, design and layout, paper and binding, and size and weight of resources contributed to visual representations for easy grasp and understanding of the concepts. Presentations are engaging, interesting, and understandable. The flow of ideas is logical, smooth, clear, and evident. Vocabulary level is appropriate to the experiences and understanding of the students. New, technical, or complex terms are strategically, clearly, and

consistently explained. The sentence length used was suitable for the target (Jamison, 2000). All of these factors help the students in meaning-making (DepEd, 2016).

Most of the jurors identified very minor errors such as conceptual, factual, grammatical, and computational, use of obsolete information, and other errors (i.e. illustrations, diagrams, pictures, maps, graphs, and tables). There is no conceptual error that may lead to the development of misconceptions or misunderstanding of the content nor factual errors which means that the content was accurate and up-to-date, and without outdated information. Metcalfe (2017) emphasizes that committing minor errors should not be exaggerated, rather it should be handled sensitively, and teachers and learners should be open to mistakes, and actively use them in becoming prepared for the test that counts. The identified minor errors can still be used as a way to facilitate new learning, to enhance memory for the generation of correct responses, to facilitate active learning, and to direct the attention of the learner while letting the teacher be aware of the focus of discussion.

Table 2 presents the summary of the jurors' evaluation along with the presence of the features – contextualization and independent learning. It is depicted that the jurors are satisfied that contextualization and independent learning features are present in the learning modules.

Table 2. Presence of the features of learning modules

Features	Mean Scores of Modules				
	Module 1	Module 2	Module 3	Module 4	Overall
Contextualization	3.68 (VS)	3.58(VS)	3.57(VS)	3.46(VS)	3.57(VS)
Independent learning	3.41(S)	3.25 (S)	3.48(S)	3.30(S)	3.36(S)

Legend: S– Satisfactory, VS – Very Satisfactory

The presence of contextualization implies that they can relate the subject matter conic sections to practical, applicable, and meaningful situations that are relevant to students (Dewi & Primayana, 2019; Lasa, Abaurrea, & Iribas, 2020). These situations are supported by facts and theories that embed the cultural, historical, ideological fabric, and/or personal experiences of students (Ampa, Basri, & Andriani, 2013; Baker, Hope, & Karandjeff, 2009). Contextualization allows the transfer of mathematics ideas (conic section) to students (Boaler, 1993; Anwar, Budayasa, Amin, & Haan, 2012). It also motivates students to recognize mathematics as part of everyday life to enhances student's ability; and deepens their understanding of all forms of mathematics to make meaningful mathematical connections (Alangui, 2017; Reyes, Insorio, Ingreso, & Hilario, 2019; Muhtadi, Sukirwan, Warsito, & Prahmana, 2017). Realistic mathematics which is supported by classroom activities constructs an understanding of meaningful mathematical ideas (Anwar, Budayasa, Amin, & Haan, 2012).

Sorsogon local information leverages students' knowledge by bringing the students' concepts from their home and community to classroom situations, allows the students to have fun, makes the lessons relevant and meaningful, resulting to lessons being effectively and efficiently delivered (Ulandari, Amry, & Saragih, 2019; Supiyati, Hanum, & Jailani, 2019; Rivet & Krajcik, 2008). The

modules assist the students to learn independently and think actively with minimal guidance and tested their mathematical resiliency (Asih, Insartoto, & Sukestiyarno, 2019). Mathematical problem-solving abilities and student learning independence increase using realistic mathematical approaches (Hasibuan, Saragih, & Amry, 2019). Students overcome their fear and appreciate the utility of mathematics if it is real and accessible (Wang, Sun, & Wickersham, 2017). Instructional support can make students' learning more engaging (Alrajeh & Shindel, 2020).

Students' Acceptability of the Learning Modules

To validate the response of the experts, the students also evaluated the acceptability of the contextualized learning modules. Table 3 presents the evaluation of the students along with the acceptability of the developed contextualized learning modules in conic sections. The table reveals that all learning modules are very much acceptable (VMA) to students along with the different indicators (Hasibuan, Saragih, & Amry, 2019).

Table 3. Summary of student acceptability of developed contextualized learning modules

Criteria	Mean Scores for Acceptability				Overall
	Module 1	Module 2	Module 3	Module 4	
Clarity	3.94(VA)	3.93(VA)	3.90(VA)	3.86(VA)	3.91(VA)
Usefulness	3.89(VA)	3.87(VA)	3.85(VA)	3.85(VA)	3.87(VA)
Suitability	3.76(VA)	3.76(VA)	3.76(VA)	3.78(VA)	3.77(VA)
Adequacy	3.84(VA)	3.84(VA)	3.85(VA)	3.85(VA)	3.85(VA)
Timeliness	3.98(VA)	3.98(VA)	3.98(VA)	3.96(VA)	3.98(VA)
Language, Style, and Format	3.83(VA)	3.81(VA)	3.79(VA)	3.80(VA)	3.81(VA)
Illustrations	3.92(VA)	3.81(VA)	3.81(VA)	3.82(VA)	3.84(VA)
Presentations	3.94(VA)	3.93(VA)	3.96(VA)	3.93(VA)	3.94(VA)
Overall	3.88(VA)	3.86(VA)	3.85(VA)	3.84(VA)	3.87(VA)

Legend: VA- Very Much Acceptable

This implies that the language and information used in the modules are presented in such a manner that it can deliver the concept of the conic section logically and independently. The consistent and purposeful use of vocabulary building greatly assisted the students (Riccomini, Smith, Hughes, & Fries, 2015). This connotes that the modules are useful to comprehensively prepare them to think logically and critically and to develop their analytical thinking and mathematical skills (Kurniati, Kusumah, Sabandar, & Herman, 2015). The use of meaningful and relevant information actively involved them in various learning activities (Alangui, 2017; Reyes, Insorio, Ingreso, & Hilario, 2019). The learning modules are suitable to address the previously identified learning gap (Rochsun & Agustin,

2020). Activities cater to unique differences, attitudes, and capabilities, with suitable prescribed topics that are relevant, interesting, and self-motivating (Singh, 2014). Sufficient and salient information is provided in the delivery of the concept. Adequate activities provided enhance their knowledge, critical thinking, skills, and attitudes. Situational problems motivate them and provide a mechanism to organize ideas. These also allow them to be reflective and to develop metacognition (Belecina & Ocampo, 2018).

The modules and their features are appropriate to be used as an urgent, tactical, transformative solution to the 21st-century educational challenges and issues (NEDA, 2017). *“The learning module is useful for the students because it can be considered as an alternative tool, and through its activities, students can exercise their mind, ability, and skills.”*, as affirmed by one of the students. Each learning module is designed and programmed from simple to complex concepts to let the learners determine the interrelationship of the concepts, and so that it will guide them to easily understand the concepts (van de Pol, Volman, Oort, & Beishueizen, 2015). One of the students emphasizes, *“[The Learning module] Provide examples that are easy to understand especially to those students who do not encounter some of the mathematical terms. Another student also stresses, well-prepared module, and for this, it can help the student to understand the topic easily.”* Furthermore, one student believes that *“the module presents real-life situations that boost students to think critically.”*

Print legibility influences the readability as it gives ease to the learner to distinguish letters and words while reading (Maile & Cooper, 2018). Indeed, the language, style, and format used in the modules aided the comprehension, understanding, and learning of the concepts in the developed material (Karimah, Hidayah & Utami, 2020; Widodo, Prahmana, Purnami, & Turmudi, 2018). The clear, simple, and relevant illustrations and presentations piqued their interest and made learning effective and enjoyable, and provided concrete visual clues (Maile & Cooper, 2018).

“The module is helpful for those students especially the student who takes Mathematics as their major that [who] did not take STEM while they are [were] in senior high school. With the help of the module, the lesson gets more exciting and it serves [s] as the bridge in a learning gap between other strand and STEM students”, one of the students affirms. This comment concluded that the developed contextualized learning modules help them bridge the learning to the subject Calculus with Analytic Geometry. These learning modules serve as supplementary materials to learn the concept independently and excite them to learn. These are affirmed by their acceptability and jurors' (experts') evaluations (Setiyani, Putri, Ferdianto, & Fauji, 2020).

CONCLUSION

The developed learning modules on conic sections for the conduct of the bridging course BSED Mathematics students who were non-STEM graduates in Senior High School were in accordance with the identified learning gap of the students to Calculus with Analytic Geometry subject. The learning modules were positively evaluated and validated by jurors' (experts' – who are Mathematics Professors). Furthermore, students' evaluations revealed that the developed contextualized learning

modules on conic sections are very much acceptable. Consequently, for further development of quality supplementary materials that promote independent learning and holistic development of critical thinking, it is highly recommended to contextualize or even localize the concepts. Contextualized learning modules can be used in the conduct of different academic remediation such as bridging courses, remote learning in times of unwanted circumstances such as pandemic (e. g. COVID-19), and among others. It is also highly recommended to test the effectiveness of the use of contextualized learning modules to the student learning outcomes.

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ASSESSMENT OF MATHEMATICS TEACHERS' PROFESSIONAL COMPETENCE

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Abstract

Development of students' mathematical skills is associated with quality teaching, which means that mathematics teachers should be able to successfully solve mathematical, teaching, and professional problems. The article aims to describe the assessment system of mathematics school teachers' professional competence, which helps identify gaps in their training and design tailor-made retraining courses. 2,359 mathematics teachers from 13 regions of Russia participated in the research on 05–29 September 2017. Foremost, we conducted a survey and collected data about their teacher category and teacher expertise. Next, we provided a preliminary diagnostic test to enable the participants to self-assess their subject matter and teaching competencies. After that, they completed a three-part diagnostic test to assess their abilities to solve mathematical, teaching, and professional problems. Finally, the participants conducted video lessons. The three-part diagnostic test and video lessons allowed determining the professional competence level for every mathematics teacher. 24% participants showed level I of professional competence, 44% – level II, 9% – level III; 23% participants did not pass the basic level of professional competence. The results show that the mathematics teachers have difficulties in solving mathematical, teaching, or professional problems so tailor-made retraining courses are required. The developed assessment system underlies designing the courses.

Keywords: mathematics school teachers, professional competence level, teaching and professional problems, diagnostic test, video lessons

Abstrak

Pengembangan keterampilan matematika siswa dikaitkan dengan pengajaran yang berkualitas, yang berarti bahwa guru matematika harus mampu menyelesaikan dengan baik permasalahan matematika, pengajaran, dan profesionalitas sebagai seorang pengajar. Artikel ini bertujuan untuk mendeskripsikan sistem penilaian kompetensi profesional guru matematika di sekolah, untuk membantu mengidentifikasi kesenjangan dalam pelatihan dan merancang kursus pelatihan ulang yang dibuat khusus untuk mereka. Sebanyak 2.359 guru matematika dari 13 wilayah di Rusia ikut berpartisipasi dalam penelitian ini yang dilakukan pada tanggal 5-29 September 2017. Kami melakukan survei dan mengumpulkan data tentang kategori guru dan keahlian guru mereka. Selanjutnya, kami menyediakan tes diagnostik awal untuk memungkinkan peserta menilai sendiri materi pelajaran dan kompetensi mengajar mereka. Setelah itu, mereka menyelesaikan tes diagnostik sebanyak tiga bagian untuk menilai kemampuan mereka dalam memecahkan masalah matematika, pengajaran, dan profesional. Terakhir, peserta membuat video pembelajaran. Tes diagnostik tiga bagian dan video pembelajaran memungkinkan penentuan tingkat kompetensi profesional untuk setiap guru matematika. Terdapat 24% peserta menunjukkan kompetensi profesional tingkat I, 44% di tingkat II, dan 9% di tingkat III; sebanyak 23% peserta tidak lulus kompetensi profesional tingkat dasar. Hasil penelitian menunjukkan bahwa guru matematika mengalami kesulitan dalam memecahkan masalah matematika, pengajaran, atau profesional sehingga diperlukan kursus pelatihan ulang yang dibuat khusus. Sistem penilaian yang dikembangkan ini mendasari perancangan kursus tersebut.

Kata kunci: guru matematika sekolah, tingkat kompetensi profesional, masalah pengajaran dan profesional, tes diagnostik, video pembelajaran

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Nowadays, it is important not only to teach the subject matter of mathematics, but also develop school students' mathematical skills. Mathematical skills development is one of teachers' main goals in

mathematics school classes. To encourage the development, a highly qualified teacher should be able to effectively apply their knowledge, life experience and expertise, as well as personal and professional values. They all contribute to quality teaching.

Comparative studies of students' learning outcomes, which have been conducted since the 1980s, show that their improvement is directly connected to improving quality of teaching (Barber & Mourshed, 2008). Mathematics teachers' quality of teaching reflects their professional competence. Raven (2002) determines the latter as a multifunctional package of continuously updating knowledge, abilities (skills) and relationships that each person requires for a full and effective inclusion in society. Kozyrev et al. (2005) specify the concept and describe it as an integral characteristic that defines ability to solve professional problems in real-life situations of professional activity with the help of knowledge, professional and personal experience, as well as values. The pan-European Tuning project determines the main components of competence: knowledge and understanding, knowledge as an action, and knowledge as the basis of value relationships in professional activity (Kennedy, Hyland, & Ryan, 2009).

School teachers' beliefs about their quality of teaching are actually reflected in their teaching activity (Kardanova & Ponomareva, 2014). It is critical that mathematics teachers should respond to students' academic needs in learning basic mathematics at schools: from arithmetic to stereometry. With this aim in mind, the teachers should be ready to efficiently carry out their professional activity. So, assessment of their professional competence and competencies as its constituents is in focus. According to Shulman (1986), a set of competencies help solve problems arising in professional activity. The problems are tools of teachers' professional training (Kozyrev et al., 2005).

The problems can be mathematical, teaching, and professional. For mathematics teachers, abilities to solve teaching and professional problems characterise their quality of teaching. The solution to teaching problems means identifying school students' errors in solving mathematical problems and their causes, work organisation and assistance in searching for relevant solutions, encouraging students to think critically and self-assess their solution. The successful solution to professional problems allows creating favourable learning conditions. The problems relate to development of school students' communication skills, enhancement of social potential, ability to regulate conflict situations, and development of self-regulation.

The solution to the problems relies upon interrelated aspects of mathematics teachers' knowledge: subject matter, teaching, and practical (Ball, Thames, & Phelps, 2008; Silverman & Thompson, 2008). The subject matter knowledge, which covers the main content of mathematics, constitutes subject matter competency. The knowledge about general and private methods of teaching mathematics describes teaching knowledge as the basis of teaching competency. The use of practical knowledge, or theories-in-use, underlies practical competency (Cheng, 2019; Rollnick & Mavhunga, 2016). Each competency can be developed at basic and advanced levels if a teacher is able to apply their knowledge, expertise, and values.

There are a number of studies which focus on assessing the effective teaching of mathematics. The assessment of a mathematics teacher's knowledge is a complicated process, since it is multidimensional. Within the research project TEDS-M, Döhrmann, Kaiser, and Blömeke (2012) explored primary school teachers' beliefs about teaching and learning mathematics effectively. They designed a TEDS-M questionnaire, which examined beliefs about the subject matter of mathematics and teaching mathematics, rather than beliefs related directly to teaching activity. There was only an investigation into the knowledge component integrated into the subject matter and teaching techniques, which is insufficient for making judgements about development of mathematics teachers' professional competence.

The Teaching and Learning International Survey centred on continuous professional development, teacher appraisal, school leadership and management, school climate, teachers' instructional beliefs, teachers' pedagogical and professional practices (Strizek, Tourkin, Erberber, & Gonzales, 2014). However, the survey did not aim to assess mathematics teachers' professional competence. Moreover, the questions in the survey did not cover the subject matter of mathematics and mathematics teaching techniques, did not provide data on mathematics teachers' professional skills. In addition, the collected data presented a subjective assessment of the respondents.

Various types of teacher knowledge, which are necessary for teaching mathematics, are identified in Chapman's article (2013). Nevertheless, the knowledge component cannot give an idea of how mathematics teachers' professional competence is developed. In turn, Koponen, Asikainen, Viholainen, and Hirvonen (2017) explain the relationship between teachers' knowledge and their qualification. The researchers developed a 72-item survey to measure teacher educators' and graduated teachers' perceptions about what the latter had learned well or poorly during their studies. Again, this research considers subjectively only the knowledge component of preservice teachers' readiness for professional activity, which is clearly not enough to assess their professional competence.

As regards teachers' knowledge assessment, the Learning Mathematics for Teaching form ("Learning mathematics for teaching," 2007) is also of interest. Orrill and Cohen (2016) used this 73-item instrument to teach proportional reasoning. The instrument measured common content knowledge and specialised content knowledge, but it did not address the activity component of professional competence.

Apart from the subject matter knowledge, mathematics teachers should implement diverse teaching practices. A sample of 644 Greek-Cypriot preservice and inservice elementary school teachers and university students with strong mathematical background demonstrated knowledge with respect to four teaching practices. They include providing and evaluating explanations; selecting and using representations; analysing student errors, misconceptions, and non-conventional solutions; selecting tasks (Charalambous, 2016).

The described studies do not equally address subject matter, teaching, and practical competencies as the main components of mathematics teachers' professional competence. The researchers tend to

focus more on the first component, while the second and especially the third one lag behind. In particular, they prioritise solutions to mathematical problems, but do not develop ability to solve teaching or professional problems.

It is important to consider the three competencies together in assessment and seek for relevant assessment tools. It is proposed to use teachers' assessment materials obtained as a result of analysing their performance review, surveys, as well as observing practical activities at competitive lessons. In her study, Smoleusova (2015) considers components of professional competence quite fully, but only according to one side of professional activity. Another approach is a formal assessment of students' learning outcomes so social aspects (for example, a student's family) strengthen or weaken development of professional competence (Huber & Skedsmo, 2016). Zee and Koomen (2016) highlight the relationship between teachers' self-assessment of their professional activity and their students' attitude towards the learning process. Kalyar, Ahmad, and Kalyar (2018) determine the relationship between teachers' motivation and their focus on mastery.

In other studies, there is an assessment with a specific tool. Temnyatkina and Tokmeninova (2018) propose the design of a teacher's performance assessment based on their students' performance. Other studies (Meuwissen & Choppin, 2015; Mohamadi & Malekshahi, 2018) create a teacher's formative assessment based on observations of their activities and identification of their performance feedback. So, testing or questioning, rather than solving problems mainly underlies the assessment. The tools reflect teachers' knowledge but do not contribute to development of their professional competence. Therefore, the research problem is: How can we assess mathematics teachers' professional competence, with subject matter, teaching, and practical competencies included? The article aims to describe the assessment system of mathematics school teachers' professional competence.

METHOD

2,253 mathematics teachers working in school grades 5–11 participated in the research, which was conducted in 13 regions of Russia on 05–29 September 2017. We aimed to determine the professional competence level (Table 1) for every mathematics teacher on the basis of a diagnostic test and video lessons.

Table 1. Characteristics of the professional competence levels

Level	Characteristics of the level
Level I	the ability to solve problems of professional activity with a predetermined condition (without considering the variability of real-life contexts)
Level II	the ability to solve problems of professional activity in a changing situation, which reflects various real-life contexts, to propose and choose various tools to solve them

Level	Characteristics of the level
Level III	the ability to solve problems of professional activity in a situation of uncertainty, which involves developing increasingly complex real-life contexts and variability of tools, as well as using new resources to solve them

At the first stage of the research held on 05–14 September 2017, we conducted a survey of research participants. We found out that they were mathematics teachers from 1,460 schools (Table 2). Their median age was 45 years old, with a different percentage of teachers in each age category: 15.60% (20–30 years old), 22.09% (31–40), 44.84% (41–55), 17.47% (56–older). 36.14% had a higher teacher category, 35.18% were awarded the first teacher category, no teacher category was characteristic of 28.68% participants. Their teacher expertise was different: 12.55% (under 5 years old), 12.15% (5–10), 20.62% (11–20), and 54.66% (over 20).

Table 2. Locations of research participants

Russian regions	Number of research participants	Number of schools involved
Volgograd region	186	13
Kabardino-Balkar Republic	196	152
Kurgan region	186	149
Leningrad region	199	155
Moscow region	191	158
Republic of Adygeya	97	68
Republic of Ingushetia	148	91
Republic of Tatarstan	204	188
Ryazan region	190	114
Tomsk region	184	91
Khabarovsk territory	182	114
Chechen Republic	216	95
Yaroslavl region	116	72

The second stage of the research provided a preliminary diagnostic test, with PDF tasks to allow the participants to self-assess their subject matter and teaching competencies. The mathematics teachers could freely access the test on the teachers' level website of Herzen State Pedagogical University of Russia (Figure 1). At that, they could solve a variety of mathematical and teaching problems by themselves and/or study the provided solutions.

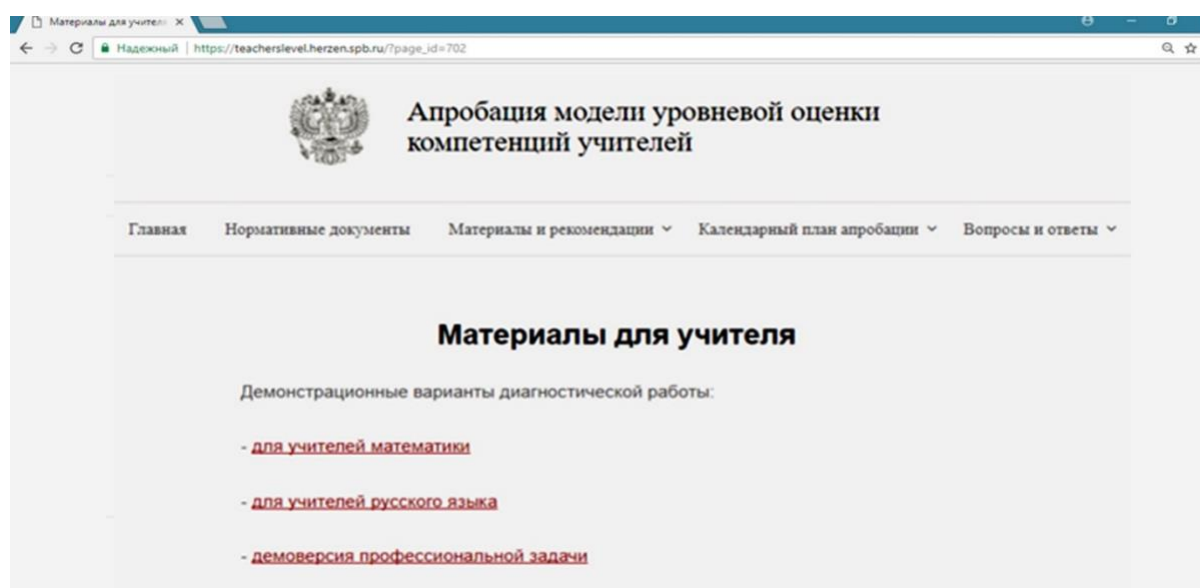


Figure 1. A webpage with a preliminary diagnostic test

Part 1 of the preliminary diagnostic test includes six basic mathematical problems, which target the subject matter knowledge about arithmetic, algebra, planimetry, and stereometry. The mathematics teachers are to solve the problems and provide a number as a single answer. There is an example of such a problem below, with the correct solution $x = 7$.

Solve the equation: $\sqrt{63 - 2x} = x$. If there is more than one solution, give the least variable x .

There are also six basic teaching problems to solve, which test teaching knowledge. While solving the problems, the mathematics teachers are involved in ordering typical of one problem, matching – one problem, and choosing single answers – four problems. There is an example of a teaching problem; its solution requires the single answer b .

Find the natural numbers x for which $2\frac{5}{9} < \frac{x}{9} < 3\frac{7}{9}$. A student's solution $x=6$ is not correct.

The reason for the error is that he/she: a) ignored the ordered set of natural numbers; b) compared only fractions in mixed numbers; c) was unable to convert the mixed number into an improper fraction; d) had misconceptions about positioning fractions on the number line; e) was unable to compare ordinary fractions.

The correct solutions to the basic mathematical problems, which are assessed 1 point maximum, allow identifying the basic level of subject matter and teaching competencies.

Part 2 comprises three advanced teaching problems to solve, which test teaching knowledge. The mathematics teachers are to provide their own scenarios of solving each teaching problem by following a set of instructions. The instructions for teaching problem 1 allow specifying school students' knowledge and skills to solve a mathematical problem, 2 – presenting difficulties in students solving a mathematical problem, 3 – providing an assessment of students' solution. At that, the teachers have to solve such mathematical problems by themselves. Here is an example of teaching problem 3.

In the equilateral triangle ABC, the altitude is 12 and the base is 32 (Figure 2). Determine the radius of the circumference.

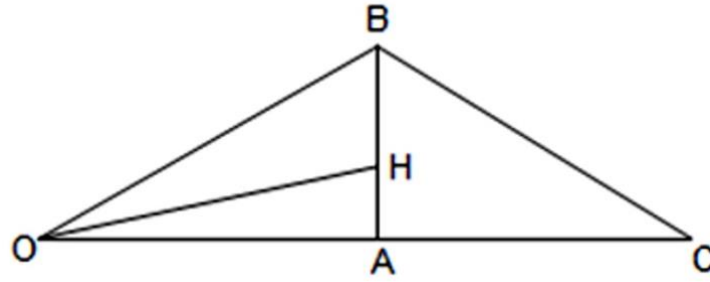


Figure 2. The equilateral triangle for problem 3 in the preliminary diagnostic test

The student solved this problem as follows:

$$0 = 256 + 144 - 24R, 24R = 400, R = \frac{400}{24} = 16\frac{2}{3}.$$

Solution: $R = 16\frac{2}{3}$.

Instructions:

1. Comment on this solution. Specify errors and flaws if there are any.
2. Give a quantitative and qualitative assessment of the solution.
3. Offer a relevant teaching technique to avoid errors in solving such problems.

The correct solutions to the advanced teaching problems help determine levels I, II, III of professional competence. The assessment of the teaching problems is based on a 5-point scale.

At the third stage of the research, the participants were to complete a diagnostic test, which we had made to identify gaps in the mathematics teachers' professional competence. The test assessed their abilities to solve mathematical, teaching, and professional problems revealed in Parts 1, 2, 3 of the tests (Table 3).

Table 3. Correlation of professional competence with the structure of the diagnostic test

Parts	Blocks	Professional competencies	Levels of professional competence
1	6 basic mathematical problems	subject matter competency	basic
	6 basic teaching problems	teaching competency	basic
2	3 advanced teaching problems	subject matter competency	I, II, III
		teaching competency	

Parts	Blocks	Professional competencies	Levels of professional competence
3	1 professional problem	subject matter competency teaching competency practical competency	I, II, III

We were elaborating all the problems covered in arithmetic, algebra, planimetry, and stereometrics from 25 May 2017 to 05 July 2017. For that, we used past tests from the standardized state exam in mathematics qualifying for the General Certificate of Secondary Education in Russia.

The diagnostic test was provided on 25–27 September 2017. The mathematics teachers had 240 minutes to complete Parts 1 and 2 of the diagnostic test, which fully correspond to Parts 1 and 2 of the preliminary diagnostic test. The teachers were not allowed to use any supplementary material to do the diagnostic test. The mathematics experts assessed these parts from 25 September to 1 October 2017.

Part 1 embraces 12 basic mathematical and teaching problems typical of those proposed in the preliminary diagnostic test. The participants can receive 1 point for each problem, which they correctly solve online on the university website. The minimum score to acquire the basic level is 10 points and above. We calculated and analysed how the participants solved each problem. The results could demonstrate if they managed to acquire the basic level of subject matter and teaching competencies. We also considered dependence of the diagnostic test results (Part 1) on the participants' teacher category and teacher expertise.

To test teaching knowledge, Part 2 contains three advanced teaching problems, with 5 points maximum for each. We assessed the participants' solution to three teaching problems and analysed the correct problem solution in accordance with some subject matter and teaching criteria, which explain the best scores (Table 4). We also compared the correct problem solution in Parts 1 and 2 of the diagnostic tests.

Table 4. Criteria for assessing advanced teaching problems

Teaching problem	Subject matter criteria	Teaching criteria
1	A well-argued answer (4 points)	A good ordering of questions and tasks (1 point)
2	The correct solution to the problem (1 point)	The correct list of errors and their causes, and revision to avoid errors (4 points)
3	The correct list of errors (2 points)	The well-grounded assessment and its correctness (3 points)

In Part 3, which was assessed 40 points maximum, the teachers presented solutions to one professional problem. Each professional problem contains a description of its conditions: a situation of a teacher's professional activity, the context that reveals the degree of uncertainty of the situation, and the problem statement. In addition to the condition's description, the professional problem is supplied with a set of instructions, which made the teachers demonstrate the problem solution. The set aligns with evaluation criteria of mathematics teachers' professional activity.

The professional problem can be the following: *When working with your students, you are faced with an urgent problem of modern teenagers: a low level of communication skills. You would like to incorporate games into your lesson plan. What school and classroom supplies can be used to enhance development of students' communication skills at the mathematics lesson?*

From 14 to 23 September 2017, the participants were to study the professional problem, solve it by following a set of instructions, and present their answer. The teachers were to show how they solved the problem on 5 pages maximum (a 12-point font, single-spaced text) and explained the answer on 10 pages maximum. They could use any relevant supplementary material at that.

On 16–29 September 2017, the mathematics experts assessed the participants' solution to the professional problem and analysed it. The assessment covered such aspects as stage-by-stage description of the problem solution, teaching material, and linguistic accuracy (Table 5), which equaled 16, 22, and 2 points correspondingly.

Table 5. The assessment criteria of the professional problem in the diagnostic test (Part 3)

Aspects of criteria	Criteria
Stage-by-stage description of the problem solution	The task to solve is well-presented.
	The task is relevant to the situation of a teacher's professional activity.
	The context, in which the situation is presented, is thoroughly described.
	The questions to answer are formulated well, specific steps to find the answer are identified.
	There is a comprehensive list of questions and relevant steps to find a solution to the professional problem in the given context.
	There is information, sources, and techniques adequate to solving the professional problem.

Aspects of criteria	Criteria
Teaching material	The information, sources, and techniques presented in a table are sufficient.
	The solution to the professional problem is presented in stages.
	There is teaching material to illustrate the solution.
	The teaching material is adequate to the solution.
	The teaching material is age- and gender-specific.
	The teaching material meets requirements of the secondary education standard.
	The teaching material is accurate.
	The solution to the professional problem is well-grounded.
	The solution prevents students from being alienated from school.
	The solution assessment is accurate.
Linguistic accuracy	There are described other situations of professional activity, where the solution can be employed.
	There are specific steps to prevent from violating the ethical code.
	There are short-term consequences of the solution.
Linguistic accuracy	The text is linguistically accurate and is logically arranged.

The teachers solved advanced teaching and professional problems in writing and uploaded their anonymous papers on the university website. They learnt about the results of each part in the diagnostic test two weeks after uploading their papers.

The fourth stage of the research, which was held on 16–29 September 2017, involved the participants conducting video lessons, with 25 students in each class.

The teachers could receive up to 38 points. Each criterion ([Table 6](#)) could be assessed 2 points maximum. The assessment of video lessons was held on 1–3 October 2017.

Table 6. The assessment criteria of a video lesson

Aspects of criteria	Criteria
Lesson topic and lesson objective	Choice of the lesson topic/objective
	Students' awareness of the lesson topic/objective
Subject matter	Subject matter knowledge
	Correlation of the subject matter with the lesson topic/objective
	Selection of relevant learning materials
	Interdisciplinary relationships
	Age-specific subject matter
	Enriching subject matter
Interaction in class	Relevance of teaching techniques
	Stimulation of students' positive motivation
	Stimulation of students' cognitive activity
	Applying a student-centred approach
	Age-specific communication with students
Development of students' self-assessment	Explanation of self-assessment
	Selection of assessment techniques
	Techniques to develop students' self-assessment
Lesson maintenance	Educational tools
	Speed of a lesson
	Friendly environment in class

Table 7 presents the matrix of points that the teachers could score after solving advanced teaching and professional problems in the diagnostic test, and conducting a video lesson. The scored points from Parts 2 and 3 of the test as well as the video lesson correspond to the teachers' professional competence level (I, II, III). The results of Part 1 are not included since they refer only to the basic level of subject matter and teaching competencies.

Table 7. The correlation of the total score with the professional competence level

Diagnostic test / Video lesson		Professional competence		
Parts	Score	Level I	Level II	Level III
Part 2	5 (teaching problem 1)	5	5	5
	5 (teaching problem 2)	≥ 3	5	5
	5 (teaching problem 3)	≥ 3	≥ 4	5
Part 3	40	≥ 21	22–31	32–40
Lesson	36	18–23	24–31	32–38
Total	91	50–59	60–76	79–91

RESULTS AND DISCUSSION

The diagnostic test results are displayed in accordance with its each part. [Figure 3](#), which illustrates a percentage of participants who correctly solved basic mathematical and teaching problems, shows a normal distribution of the points scored in Part 1. Mathematical problem 6 proved to be the most difficult to solve as only 20.10% participants managed to use methods in number theory correctly. However, mathematical problem 3 was the easiest one, with 91.30% involved to identify and employ mathematical facts in order to answer the provided question.

The smallest number of participants (28.99%) succeeded in solving teaching problem 7, which assessed basics of teaching mathematics. Only 54.37% coped with the least difficult teaching problem 12, which tested the mathematical subject matter knowledge as well as skills to select, interpret, and employ mathematical facts relevant to teaching activity.

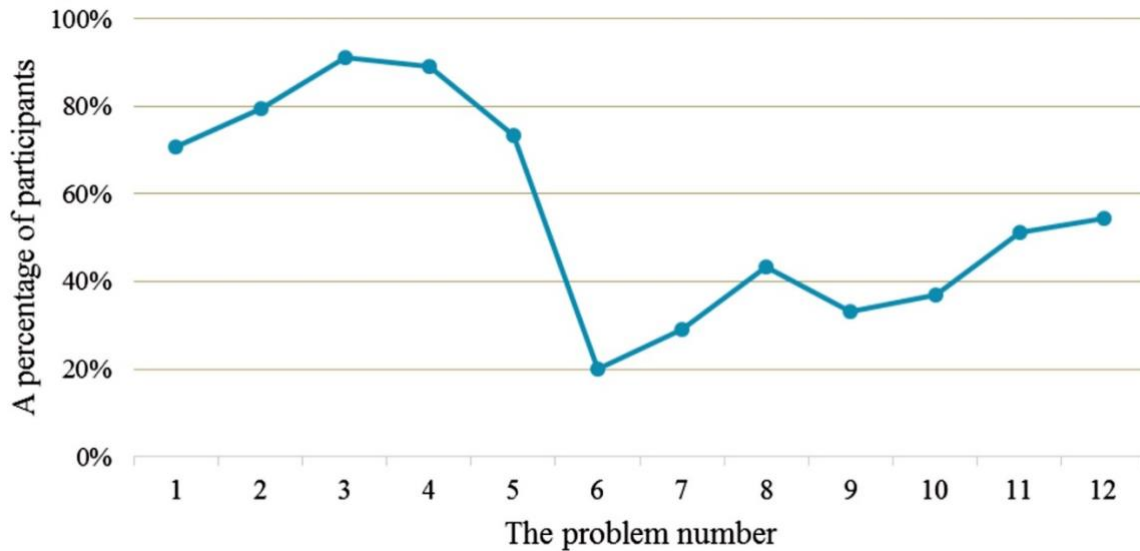


Figure 3. The results of the correct problem solution in the diagnostic test (Part 1)

Figure 4 correlates a percentage of participants with the number of total points gained in Part 1. It demonstrates a lower basic level of teaching competency compared to subject matter competency. 19.20% participants gained 8 points, 17.94% – 7, 15.99% – 6, while the highest score (12 points) was found only among 0.39% participants, the lowest one (0 points) – 0.44%. The median score was 7.

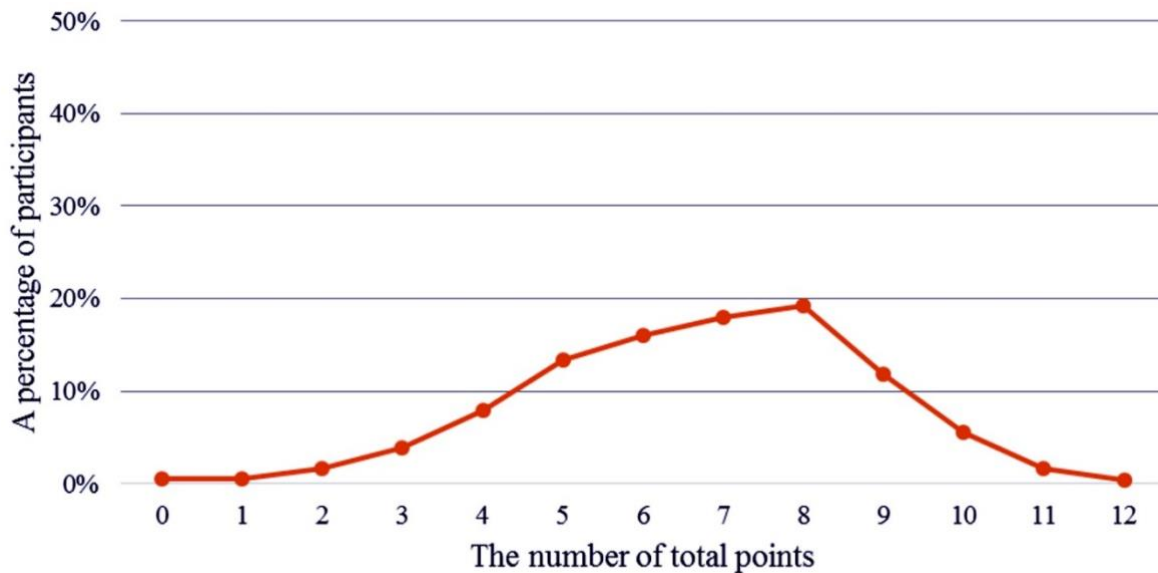


Figure 4. The total results of the diagnostic test (Part 1)

We compared the correct problem solution among the participants with different teacher categories (Figure 5) and concluded that a higher teacher category allowed them to receive higher results. 26.10% correctly solved the most difficult basic mathematical problem, 96.90% – the easiest one. Correspondingly, there were 17.57% and 91.41% with the first teacher category, 15.63% as well

as 84.05% – with no teacher category. As regards the most difficult and easiest basic teaching problems, the results were as follows: 53.10% and 31.40% among those with a higher teacher category, 55.09% and 32.23% – the first teacher category, 55.10% and 21.88% – no teacher category.

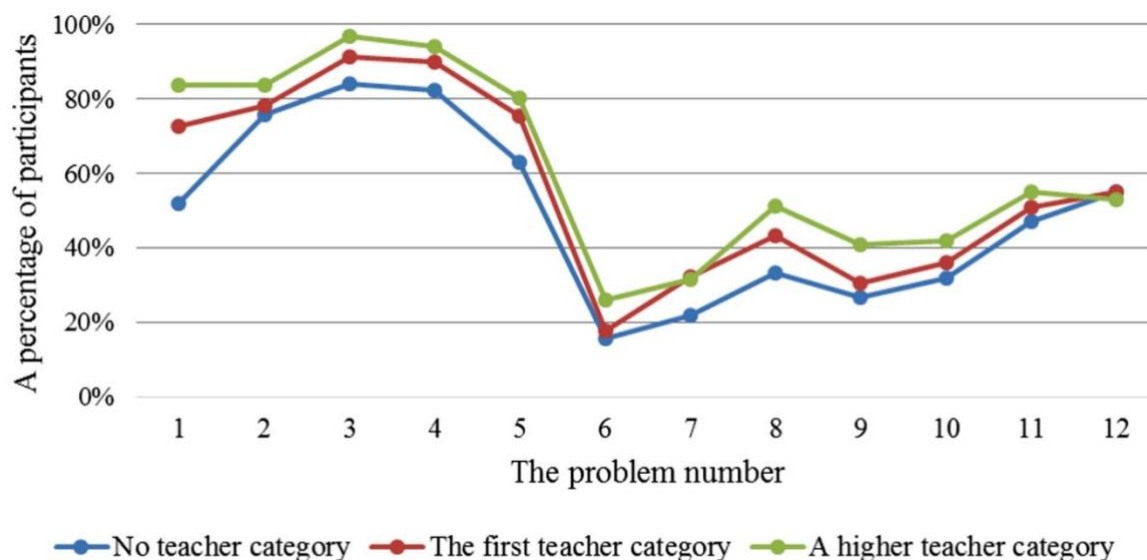


Figure 5. The results of the diagnostic test (Part 1) related to the participants' teacher category

The participants' teacher expertise influenced the correct problem solution but not so significantly (Figure 6), which is evident in solving the most difficult and easiest problems. On average 22.05% participants with longer teacher expertise (over 10 years) managed to solve the most difficult basic mathematical problem, unlike 15.33% of those with shorter teacher expertise (under 10 years). The average percentage of teachers who succeeded in solving the easiest mathematical problem was 92.98% and 84.18% correspondingly. There are no striking differences in solving basic teaching problems since the success did not largely depend on the teacher expertise. The percentage of teachers (53.63% and 31.43%), who had been working for over 10 years, was similar to 54.45% and 22.54% with teacher expertise of under 10 years, as they relate to solving the most difficult and easiest teaching problems.

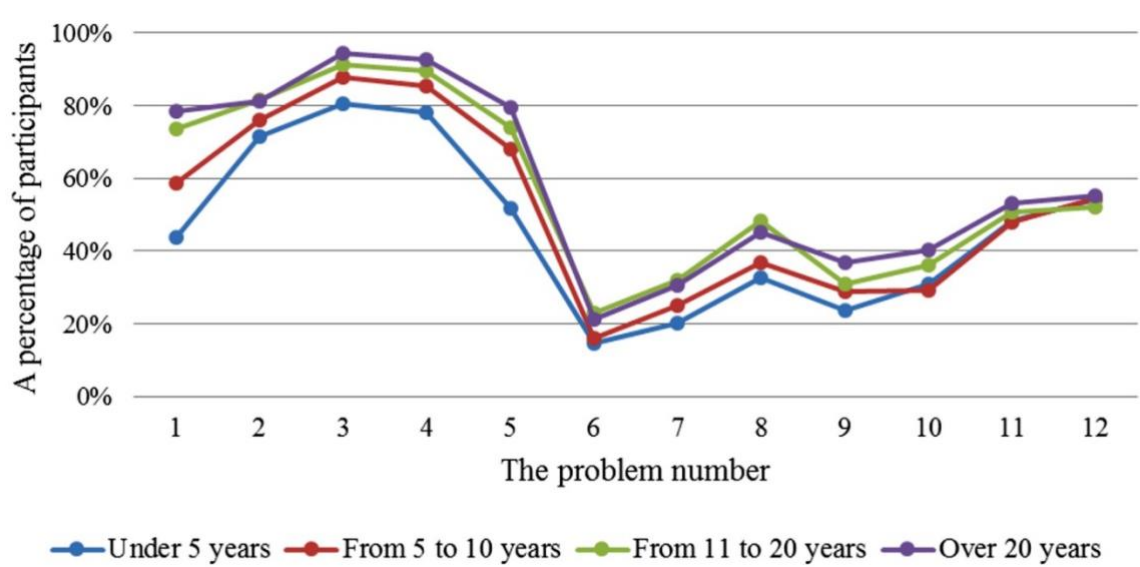


Figure 6. The results of the diagnostic test (Part 1) related to the participants' teacher expertise

Figure 7 demonstrates how a percentage of participants, who correctly solved three advanced teaching problems provided in Part 2, correlates with the total score gained. The median score was 8 points out of 15.

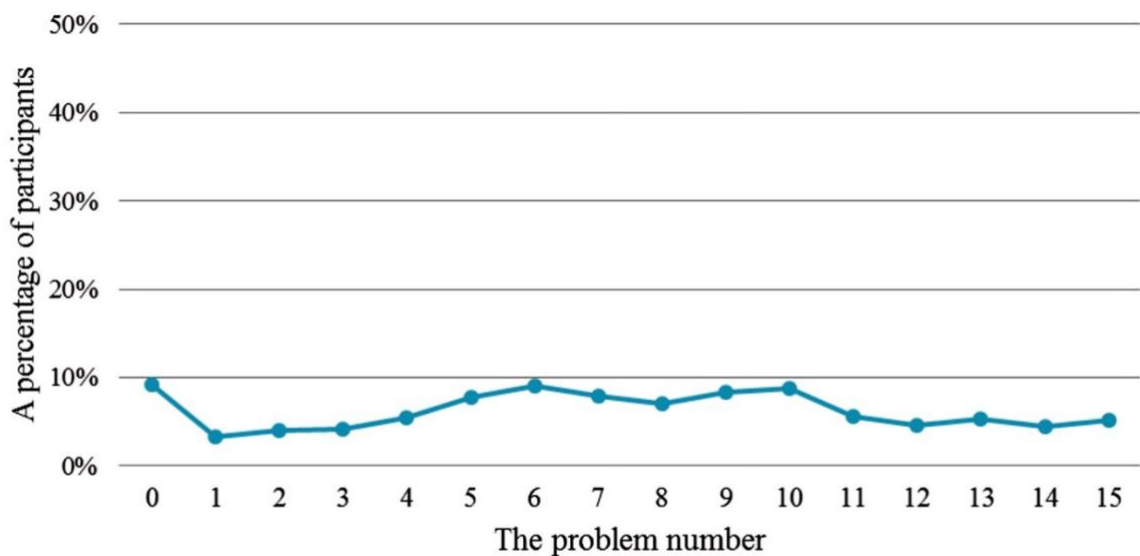


Figure 7. The results of the correct problem solution in the diagnostic test (Part 2)

Figure 8 reveals how many points the participants received for solving each advanced teaching problem in accordance with their percentage.

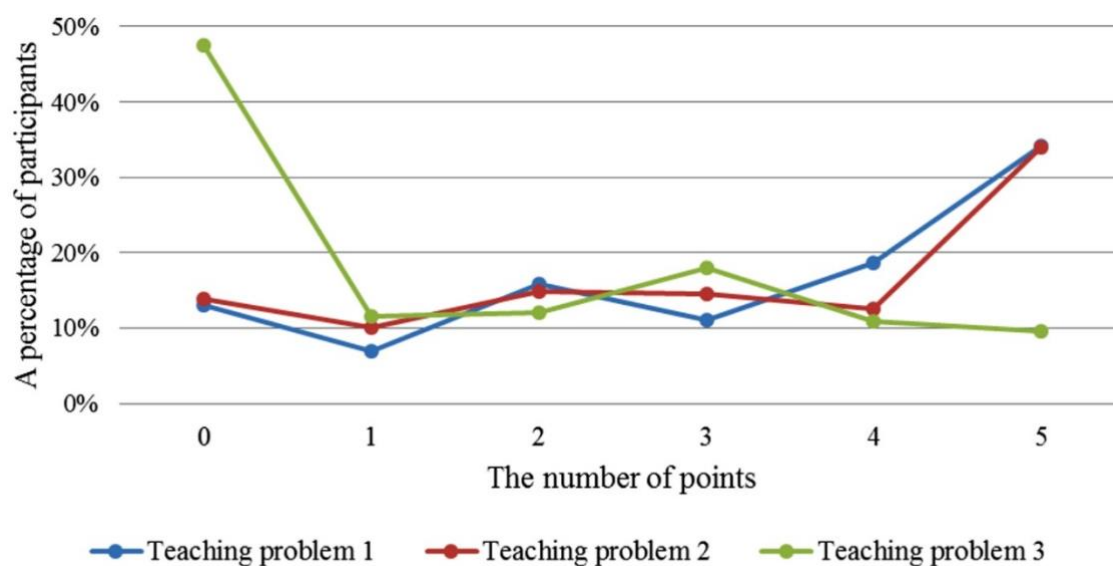


Figure 8. The total results of the diagnostic test (Part 2)

The results of solving teaching problem 1 correctly highlight some difficulties. Firstly, a half of the participants (49%) failed to correctly solve the advanced problem. Secondly, less than a half (45%) were able to compose a sequence of questions and tasks for the students, who sought to find a solution to this problem. The results of solving teaching problem 2 showed that 14% participants were able to identify all the errors and their causes, but could not explain how to organise their students' work to eliminate them. 44.7% of them were able to partially or fully propose a teaching technique to eliminate them. The obtained results relevant to teaching problem 3 demonstrated a low percentage of participants (19.6%) who successfully solved it. They could not correctly evaluate their students' work on solving a mathematical problem.

In Figure 9 we can see that that the participants, who scored higher points for solving the teaching problems in Part 2, showed higher results in Part 1 of the diagnostic test. Still, these participants (about 6%) scored less than 6 points, and only about 16% acquired the basic level.

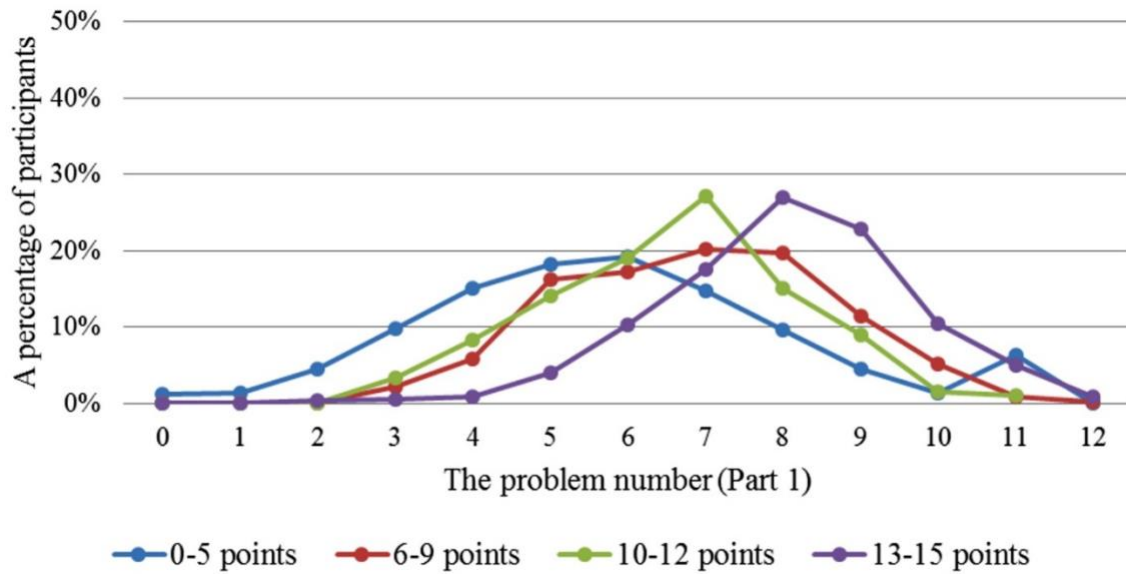


Figure 9. The total results of the diagnostic test (Parts 1 and 2)

In Figure 10, there is a correlation between the median and maximum points gained in accordance with the assessment criteria of the professional problem in the diagnostic test (Part 3).

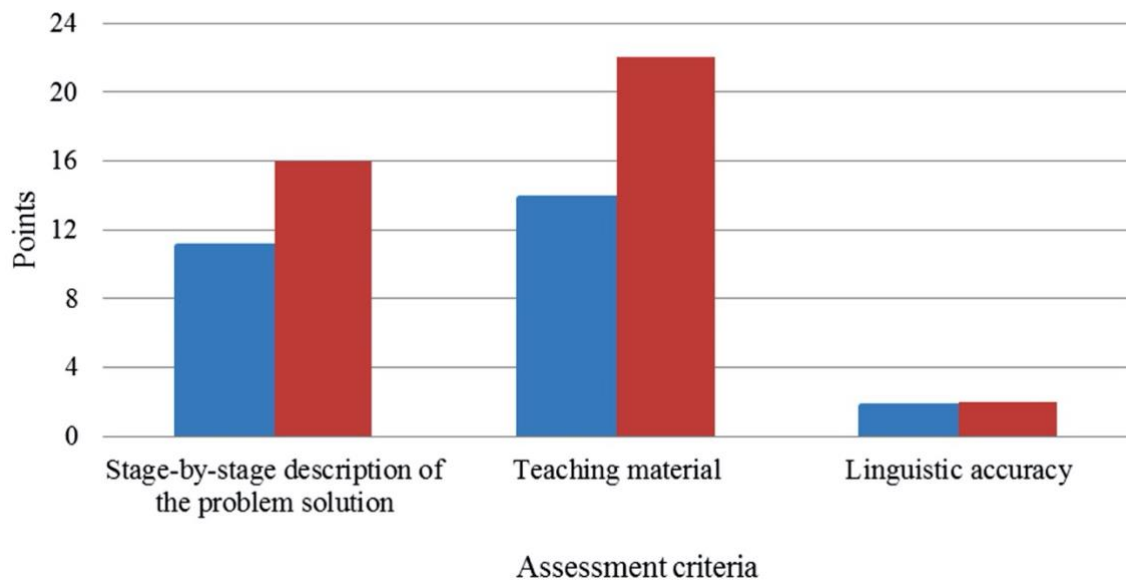


Figure 10. The median results in the diagnostic test (Part 3)

Evidently, the participants had difficulties in describing the solution to the professional problem stage by stage. Only 14% mathematics teachers managed to gain the maximum total score of 40 points, which indicates their success in completing Part 3. 19.6% teachers failed to receive the minimum total score of 21.

Figure 11 correlates the median with maximum points that the mathematics teachers gained when conducting their video lessons. The median score proved to be 25.9, out of 38 maximum points.

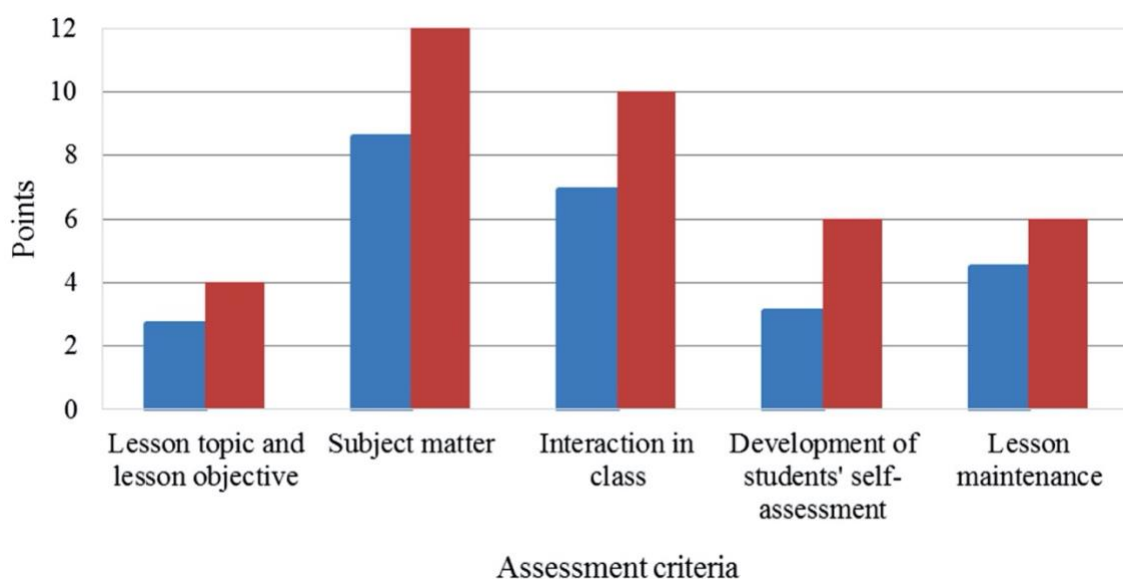


Figure 11. The median results of conducting video lessons

The total results of the diagnostic test and conducting video lessons indicated that 23% mathematics teachers did not pass the basic level of professional competence. 24% participants showed level I of professional competence, 44% – level II, 9% – level III.

The results of the diagnostic test confirm that the mathematics teachers' basic level of subject matter competency grows with the teacher category and teacher expertise. However, the correct solution to basic teaching problems, which aimed to assess the basic level of teaching competency, remains insufficiently high even among the mathematics teachers with a higher teacher category and longer teacher expertise. In addition, the success in solving some problems does not always depend on the teacher category and teacher expertise.

Some difficulties in solving advanced teaching problems suggest that not enough attention is paid to organisation of school students' work on mathematical problems. In fact, this work is an essential part in recognising students' knowledge and skills to solve a mathematical problem, which are known as students' mathematical literacy skills (Prabawati, Herman, & Turmudi, 2019). In this regard, pedagogic efficacy can contribute to students' achievement in performing mathematical operations (Fung et al., 2017). Additionally, not every teacher was able to explain how to successfully organise students' work to eliminate errors in solving mathematical problems. Such work may involve tasks to help students understand mathematical concepts in depth, rather than memorise formulae (Halimah, Subanji, & Septi Nur Afifah, 2019). Another drawback was the teachers' inability to effectively assess students' solutions. For instance, they can find wrong solutions to be correct or may not be aware of

errors in solutions (Mokh, Othman, & Shahbari, 2019). In this case, comparative judgement may be advantageous (Jones & Inglis, 2015).

The analysis of the presented solutions to the professional problem enabled to make some inferences. In particular, the majority of mathematics teachers exhibited a fairly high willingness to work with the subject matter. It can become the basis of implementing motivation as the main condition to improve the quality of teaching mathematics (Timerbaeva, Fazleeva, & Shakirova, 2019). Moreover, the majority of teachers pinpointed significance of the professional problem, rather than referred to the context where the situation of professional activity could be found.

Nevertheless, some teachers misunderstood the real-life professional context, which affected acquisition of practical competency. The participants' failures in solving professional problems are associated with their formal approach to the problem solution, without utilising their expertise. At times, the teachers could not fully grasp the content of the professional problem so their answer lacked a well-elaborated description of the problem solution. In addition, the teachers failed to follow the instructions to solve the professional problem consistently and accurately, or they demonstrated solutions to several problems instead of one. Moreover, the use of Internet sources to provide the problem solution or presumably copying other teachers' answers resulted in similar descriptions of such solutions. These are the reasons why the mathematics' teachers were not good at implementing their professional competence.

The results of conducting video lessons demonstrate an absence of providing the lesson's specific objectives as a common problem that the teachers faced. There are some striking differences between the median points, which are significantly lower especially for developing students' self-assessment, and maximum ones. The differences illustrate the teachers dominating in class, rather than creating favourable conditions to encourage their students' independent work.

The total results pinpointed some professional difficulties arising in solving teaching and professional problems. They resulted from the teachers' inability to solve practical tasks of constructing the educational process. In particular, they found it difficult to organise their students' work on mathematical problems, search for their solutions, and evaluate their students' activities. In classes the teachers were not capable enough of implementing a study programme in mathematics since some of them neglected real-life conditions of professional activity to support student progress.

It is important to eliminate barriers to teaching competency acquisition so formative assessment can assist. Alongside with professional problems, performance measures can help assess practical competency (Howell, Stone, & Kane, 2019). The measures are associated with students' abilities to perform in a variety of assessment tasks (Poh, Muthoosamy, Lai, & Hoe, 2015).

The current research makes a significant contribution to the training of mathematics teachers due to the horizontal and vertical structure of professional competence. The interrelated structural components (subject matter, teaching, and practical competencies) reveal the horizontal structure, whereas levels I, II, III refer to the vertical one. Therefore, we developed tools to assess each component

both vertically and horizontally. In particular, we elaborated teaching and professional problems that allow identify in gaps in the professional training of mathematics teachers. They enable to design tailor-made retraining courses aimed at developing mathematics teachers' professional competence.

CONCLUSION

The article describes the assessment system to identify gaps in mathematics school teachers' professional competence and reveals the results of its implementation, which enable to design tailor-made retraining courses. The research findings show that the mathematics teachers with a higher teacher category and longer teacher expertise succeeded in conducting a video lesson as well as passed the basic level of the diagnostic test and obtained higher scores. The findings prove that the provided assessment system corresponds to the aim of the research.

We assessed the mathematics teachers' level of professional competence after they had completed a three-part diagnostic test and conducted video lessons. These activities proved to be adequate to real-life situations of professional activity since mathematics teachers regularly deal with mathematical, teaching, and professional problems in mathematics school classes.

The teachers did not find mathematical problems difficult to solve, unlike basic and advanced teaching problems. Advanced teaching problems required knowledge about which teaching techniques to use in order to help students solve mathematical problems and eliminate any possible errors they could make. Some mathematical teachers failed to use the techniques properly and could not cope with the effective assessment of students' solutions to mathematical problems. On the other hand, professional problems were the most difficult to solve due to the fact that most teachers did not consider real-life conditions of professional activity to support students' progress in solving mathematical problems.

The total scores of the diagnostic test and conducting video lessons were attributed to different levels of professional competence: 24% participants acquired level I, 44% – level II, 9% – level III. 23% did not pass the professional level assessment. These results explain the need to offer a tailor-made professional retraining programme for each mathematics teacher, which meets their needs. To master solving mathematical, teaching, or professional problems in tailor-made retraining courses is likely to improve the professional competence level.

A minor focus on video lessons serves as a limitation. It is planned to describe in detail peculiarities of the lesson topic and lesson objective, subject matter, interaction in class, development of students' self-assessment, and lesson maintenance as they relate to different teacher categories and teacher expertise. In addition, the analysis of video lessons may show how conducting lessons contributes to acquisition of the mathematics teachers' professional competence.

When analyzing the results, we considered only the teacher category and teacher expertise, but ignored other important aspects of the mathematics teachers' professional activity. This major limitation can be mitigated if we analyze some other aspects. They are the teachers' place of residence, the number

of students at school, experience of teaching mathematics outside the school, experience in organizing teaching, the volume of the teaching load, experience in preparing students for the All-Russian Olympiads, methods used to monitor learning outcomes, participation in extracurricular activities, applying information communication technologies, use of teaching technologies.

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AN ANALYSIS OF LEARNERS' SOLUTION STRATEGIES IN THE CONTEXT OF MODELLING TASKS

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Abstract

It remains a challenge for teachers to integrate modeling tasks in everyday mathematics classes. Many studies have been conducted that show the difficulties faced by teachers. One of the challenging aspects in this regard is that of assessment. In the present study, a connection between structures of learners' solution strategies and cognitive considerations is established to develop a practice-oriented instrument to determine and assess the complexity of solution strategies of modeling tasks. In this paper, the selected learners' strategies' structure was analyzed in-depth to identify the underlying cognitive structure. The results show that thought operations carried out in parallel complicated a solution strategy. However, the results also support a purely sequential thought operation approach without weighting parallel thought operations, which corresponds to an intuitive assessment procedure by mathematics teachers. As assessment is a great challenge for many teachers in the context of modeling tasks, this study provides a promising frame of reference for further research in this important domain of assessment and modeling.

Keywords: modelling tasks, cognitive structure, solution strategies, mathematics education

Abstrak

Ini merupakan suatu tantangan bagi guru untuk mengintegrasikan tugas pemodelan dalam kelas matematika sehari-hari. Banyak penelitian telah dilakukan yang menunjukkan kesulitan yang dihadapi oleh para guru. Salah satu aspek yang menantang dalam hal ini adalah penilaian atau evaluasi. Dalam penelitian ini, hubungan antara struktur strategi penyelesaian peserta didik dan kemampuan kognitif dibangun untuk mengembangkan instrumen berorientasi praktik untuk menentukan dan menilai kompleksitas strategi penyelesaian tugas pemodelan. Makalah ini membahas tentang struktur strategi peserta didik yang dipilih, dianalisis secara mendalam untuk mengidentifikasi struktur kognitif yang mendasarinya. Hasilnya menunjukkan bahwa operasi pemikiran yang dilakukan secara paralel mempersulit strategi penyelesaian. Namun, hasil ini juga mendukung pendekatan operasi pemikiran sekuensial murni tanpa pembobotan operasi pemikiran paralel, yang sesuai dengan prosedur penilaian intuitif oleh guru matematika. Hal ini dikarenakan penilaian atau evaluasi merupakan tantangan terbesar bagi banyak guru dalam konteks tugas pemodelan, penelitian ini memberikan kerangka acuan yang menjanjikan untuk penelitian lebih lanjut dalam domain penting penilaian dan pemodelan ini.

Kata kunci: soal-soal pemodelan, struktur kognitif, strategi-strategi penyelesaian, pendidikan matematika

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This paper addresses the observation that mathematical modelling tasks generally do not feature prominently in everyday school life. As teachers are directly responsible for the implementation of educational recommendations and standards, such as incorporating modelling tasks in the teaching of mathematics, they are the key agents of change in ensuring that this happens (Fernandes, 1995; Wilson & Cooney, 2002). Mathematical modelling is cognitively demanding, for both students and teachers (Blum & Ferri, 2009) and implementing modelling in school is thus a great challenge for both students and teachers (English, 2009; Rivera, 2018; Warwick, 2007). Besides the organizational and material-oriented challenges inherent in using modelling tasks, other problems teachers face relate to the open-

ended nature of these modelling tasks. Open-ended tasks lack predictability and make use of a variety of solution strategies, thus complicating assessment (Schmidt, 2010). This requires teachers to adopt a pedagogy that reflects this openness to working with different solution strategies.

This paper reports on a bigger research study which specifically focused on a particular method to determine the complexity of solution strategies of modelling tasks. It is hoped that the findings of this research might lead to easier manageability of modelling tasks and can also be a starting point for alternative ways of assessment in the teaching of modelling tasks. Therefore, the study considered the following main question: does a thought structure analysis, together with its underlying cognitive and structural assumptions, lead to a suitable characterization of the complexity of solution strategies to modelling tasks?

When assessing the solution to mathematics tasks, their numerical value is often taken as the dominant indicator of a right or wrong solution. When dealing with modelling tasks, however, it is often not meaningful to merely assess a solution's numerical value due to the multiple solution nature of these modelling tasks. A common observation is that modelling tasks cannot be assessed as objectively as traditional task formats (Spandaw & Zwaneveld, 2010). Studies show that assessment of modelling tasks is a major challenge, which often hinders mathematics teachers from integrating modelling in their mathematics classes (Jensen, 2007). The complexity of the solution is a critical measure when assessing student solutions (Shanta & Wells, 2020). Therefore, if we want modelling to be part of mathematics teaching, ways of practical assessment need to be found. This study was thus undertaken to determine the complexity of solution strategies to selected modelling tasks, which can then form the starting point for further assessment considerations.

The Role of Cognitive Structures

A common procedure when solving mathematical problems in general is what Bourbaki (1961, p.163) refers to as “recalling structures”. In school mathematics learners are often taught to structure a solution in terms of “a question”, “calculation” and “the answer”. Those structuring procedures are meant to relieve and support the learners' thinking processes and help to objectivize the individual solution strategies. The latter is closely connected to assessment practices where structures also play an important role. A common assessment procedure that mathematics teachers often use is to provide a sample solution in advance and to identify the necessary intermediate steps to attain a solution which is then scored. Because solutions to modelling tasks are often open-ended and can vary widely, there is, however, no unique sample solution. This complicates an objective assessment of solutions.

The common, mostly intuitive, assessment procedure mentioned above has similarities with structural considerations (e.g. the use of arithmetic trees, Figure 1) within the field of word problems. Breidenbach (1969) looked at the structural-substantial complexity of a word problem among other things, to determine its complexity. For example, in the task illustrated in Figure 1 there are three variables in the first step, where the third variable is uniquely determined from the other two by one

operation (in this case it is multiplication). Breidenbach called this arrangement “Simplex”. A linking of several Simplexes in the solution process is called a “Komplex”. Further developments by Winter and Ziegler (1969) lead to the arithmetic tree illustrated in Figure 1, which is still used in mathematics textbooks today.

An obvious, but so far empirically not validated conclusion, is that a large number of Simplexes and a complicated nesting of them, have an effect on the complexity of the tasks' solutions. Cohors-Fresenborg, Sjuts, and Sommer (2004) describe task complexity, inter alia, by the criterion “cognitive complexity” of thought processes. This allows for simultaneity or nesting of what we call “thought steps”. In this context Kaune (2000) also distinguishes between two aspects of cognitive complexity. These are firstly the processing of multiple information within one thought step, and secondly the influence of previous information and data.

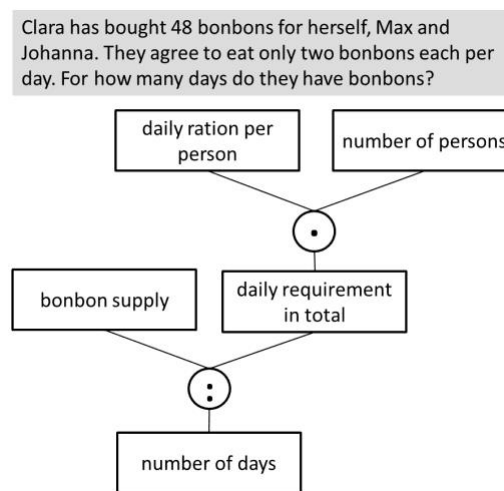


Figure 1. An example of an arithmetic tree for a selected task according to Winter and Ziegler (1969)

Within the modelling discourse, cognition and assessment of thinking has particular relevance to study how students deal with modelling tasks. Blum and Ferri referred to students' difficulties with modelling tasks by the tasks' cognitive demand (Blum & Ferri, 2009, p. 46). Tanner and Jones investigated the influence of (meta)cognitive skills on modelling (Tanner & Jones, 1993; 2002) and assessed, inter alia, cognitive ability.

Cognitive Complexity

The present study investigates the cognitive complexity of a solution strategy by analyzing the structural complexity represented by its arithmetic tree-like structure as illustrated in Figure 1. The coherence of structural considerations and cognitive theories play an important role in solving modelling tasks. In their study, Fletcher and Bloom (1988) assumed that text comprehension is an important problem-solving process in itself, where the problem solver must find a causal chain to link the start to the end of the text. In order to form these links, Fletcher and Bloom (1988) posited that the

information concerning the task must be kept in the mind of the problem solver. Results of their study show that it is important for the problem solver to access the immediately preceding information before proceeding to the next step in the causal chain. It can be concluded that the role of the working memory is to keep key information available. This is necessary to link old and new information (Baumann, 2000). By relating these findings to structural considerations of a particular solution strategy represented as an arithmetic tree, statements can be made about its cognitive complexity. On the one hand, an arithmetic tree-like structure (Figure 1) can be interpreted as a causal chain since the start (the given information in the task text) and end (the solution of the task) is linked by chain links. In the context of solving mathematics tasks we refer to these links as intermediate steps in the solution process. On the other hand, the immediately preceding information can be identified as relevant intermediate steps. Thus, it can be concluded that the immediately preceding intermediate step must be kept active in the working memory to master the next step in the chain. The assumption that mental processing capacity is limited (Sweller, 1988), leads to the conclusion that a plethora of information which has to be kept active at the same time, can complicate the solution process. Thus, it can be concluded that the working memory load is dependent on the number of intermediate steps necessary to master the current intermediate step in the problem-solving process. This means that the load of the working memory increases with the increasing amount of information needed at various points in the solution process.

Based on these theoretical considerations the following definition of thought structure was taken fundamental for the present study (Reit, 2016, p. 58):

“A thought operation is a necessary structuring process for finding a solution, which results directly (i.e. without intermediate calculations) from one or several previous data. Previous data are either (intermediate) results from previous thought operations or given data from the tasks’ text.”

A thought structure thus illustrates the logical sequence of thought operations (Reit, 2016, p. 60). The present study establishes a framework to structurally analyze student solutions to modelling tasks with the aim of determining their complexity. It is assumed that the number of sequential and parallel thought operations provides information about the cognitive complexity of a solution strategy.

METHOD

This study is framed by three main phases (see Figure 2 for phase one and two). In phase one an empirical study was conducted with 1800 student solutions to selected modelling tasks. The different solution strategies were identified. In phase two of the study, a so-called thought structure analysis (Reit, 2016) was conducted. Here the structures of the various solution strategies were analyzed by identifying their specific thought operations. At this point the number of parallel and sequential thought operations were identified. To translate this data quantitatively, statistical models were developed. Some of these models differentiated between sequential and parallel thought operations by applying a

weighting to the latter. Of particular interest were the differences in the models that applied or did not apply a weighting to parallel thought operations, as we wanted to determine which model could most appropriately identify the complexity of the solution strategies of modelling tasks. In a third phase the evaluation of the complexity models took place. The theoretical complexity of the solution strategies was statistically compared to the average score of the student solutions as a measure for the empirical complexity. This was based on a rubric. In particular, we analyzed which complexity model led to a statistically verifiable and substantially reasonable coherence between complexity and score in terms of “high score, low complexity” and vice versa. These results finally yielded information about the influence of parallel thought operations on the complexity of the solution strategies.

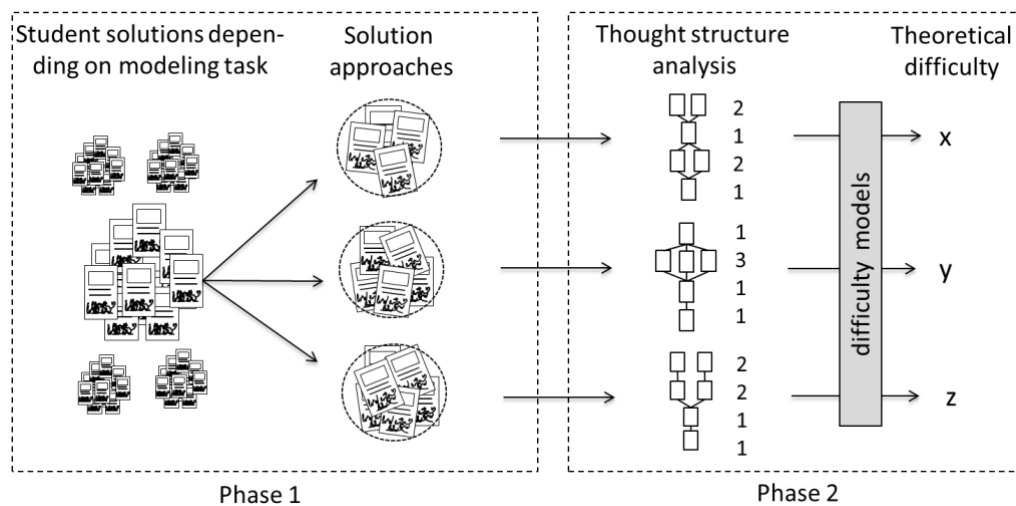


Figure 2. Method in phase 1 and phase 2 of the study

For the bigger study referred to above, five modelling tasks were developed according to predefined criteria (see Reit & Ludwig, 2015a; 2015b). Three of these five tasks were randomly selected and published in a booklet which was distributed to the 1800 Grade 9 students (15 years of age) from German grammar schools. Each student was asked to solve the three tasks individually. Each student had 45 minutes to complete the booklet. For the purpose of this paper we will discuss “the Potato problem” as illustrated in Figure 3. The students were asked to mathematically estimate the number of French fries which can be cut out of a single potato.

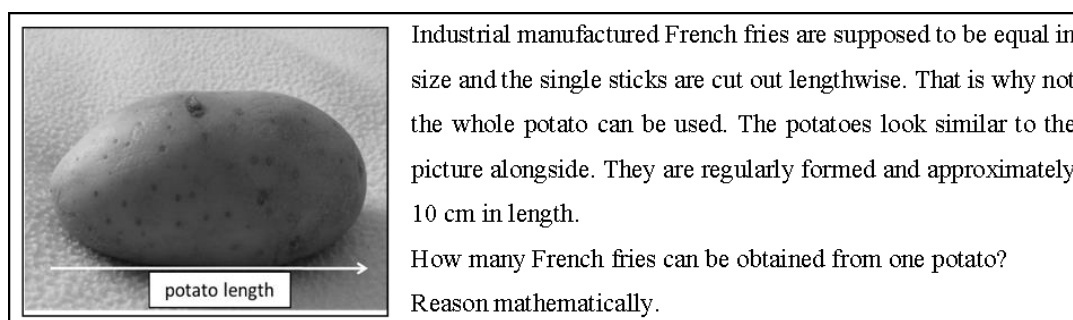


Figure 3. The potato modelling task

Thought Structure Analysis

In phase one of the bigger study (Figure 2) all student solutions were clustered into different solution strategies according to which mathematical model was used in the solution process (for a detailed discussion of the different solution strategies see Reit, 2016) (interrater reliability: Cohen's $\kappa \in [0.702-0.949]$). In a second step, each solution strategy was analyzed to reveal its thought structure - see Figure 4 for the Potato modelling task. In particular, single thought operations were identified and a thought structure was set up. This provided information about the number of parallel and sequential thought operations that took place within the respective solution strategy.

Figure 4 illustrates a particular student solution to the potato modelling task, representing the solution strategy layer and its corresponding thought structure. The potato and the French fry were approximated to and represented by cuboids. By calculating the number of French fries fitting into height and width of the potato and multiplying these results, the total number of French fries could be mathematically estimated (Figure 4). The thought structure illustrates the thought operations needed for the next step in the solution process. This can be represented as a vector representation ((3,1,1) for the solution strategy in Figure 4), indicating the number of parallel thought operations in each thought structure level. So, in this case there are 3 thought structure levels which have to be mastered consecutively to come to a final solution. In the first level there are 3 thought operations to be linked together mathematically correctly to come to the second level with one thought operation, and in the third level there is also one thought operation to be mastered.

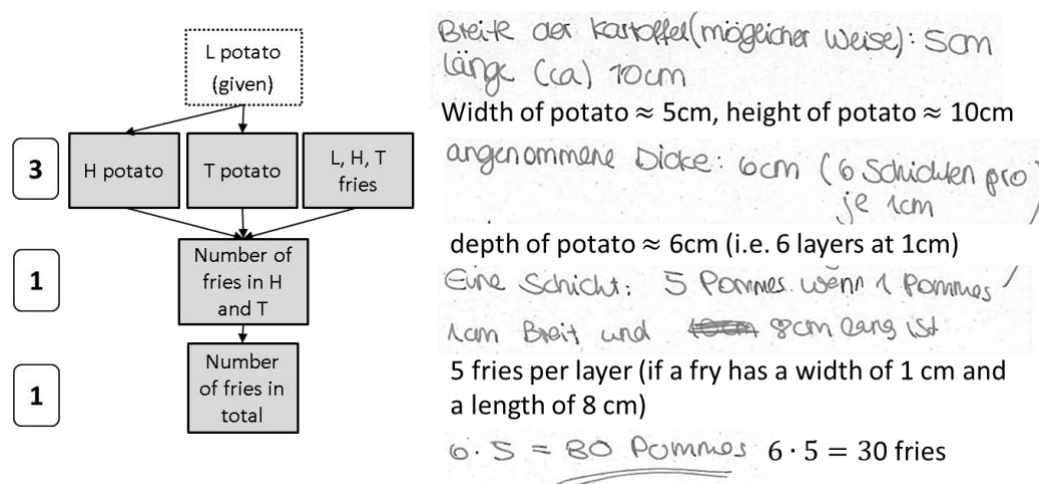


Figure 4. Student solution (solution strategy layer) (right) together with its thought structure (left), indicating the number of parallel thought operations per level

It is recognized that besides the cognitive aspects, linguistic formulations also play a role when characterizing the complexity of a modelling task (e.g. Cohors-Fresenborg, Sjuts, & Sommer, 2004; Walzebug, 2014, p. 161). Since these aspects are not considered by the thought structure model explained above, the complexity of the task text is integrated as a separate factor (see Reit 2016, pp.

103). A scheme following Cohors-Fresenborg et al. (2004) was developed by Reit (2016, p. 104) which classifies the complexity of the tasks' text and adds to the cognitive complexity of a student solution.

Complexity Models

The empirical complexity represents the average (normalized) score of the student solutions being clustered to the respective solution strategy. The thought structures currently provide a vectorial representation which we subsequently translated into a scalar. The question arises whether parallel thought operations complicate a solution strategy. Several models (Table 1) were used to translate the vectorial representation of the number of parallel and sequential thought operations into a scalar value. They differ in the kind of weighting of the parallel thought operations. The aim of determining a theoretical complexity was not primarily to find the best possible fit of a possible complex complexity model, but rather to find a model that could be used in school practice, and that, at the same time reflected the data well.

Table 1. Different Complexity Models Applied to the Potato Modelling Task

Thought Structure Vector	(3,1,1)	Theoretical Complexity
Addition model	$3+1+1$	$= 5$
Maximum model	$\max(3,1,1)$	$= 3$
Norm model	$\sqrt{3^2 + 1^2 + 1^2}$	≈ 3.3
Quadratic model	$3^2 + 1^2 + 1^2$	$= 11$
Factorial model	$3!+1!+1!$	$= 8$

The *addition model* in Table 1 simply adds up the number of thought operations which results in a theoretical complexity of 5 for the solution in Figure 4. It does not differentiate between parallel and sequential thought operations. This, we argue, is possibly the most widespread assessment procedure of mathematics teachers: scoring of intermediate steps within a solution and subsequent addition of these partial scores. The results of the addition model then show whether this so far intuitive assessment procedure can be empirically supported and thus transferred also to modelling solutions. The *maximum model* refers to the maximum metric of a vector. Using this model, the theoretical complexity is determined by the thought structure level with the highest number of parallel thought operations.

An obvious way to translate a vector into a scalar is to calculate its norm. This is referred to as the *norm model*. This leads to a quadratic weighting of parallel thought operations as is also done by the *quadratic model*. Another possibility of operationalizing the theoretical complexity is based on the consideration that the sequence of processing thought operations on the same level in the thought structure is arbitrary. This content-related notion is reflected by the *factorial model*. Following the factorial model, parallel thought operations are seen to be cognitively more demanding and thus result in a more difficult solution strategy than sequential ones.

This theoretical complexity was then compared to an empirical complexity representing the average (normalized) score of the student solutions being clustered to the respective solution strategy. For this purpose, a rubric was set up based on the thought structures of the solution strategies so that each thought operation corresponded to an essential intermediate step of the solution. Identifying and assessing essential intermediate steps and scoring them corresponds, we argue, to the common practice for evaluating written student solutions in mathematics. Depending on whether a thought operation was executed completely correctly, with errors, or not at all, 1, 0.5 or 0 credits were awarded. Follow-up errors were considered, as calculation errors should not be excessively weighted.

RESULTS AND DISCUSSIONS

In this section a comparison is drawn between theoretical difficulty, resulting from the thought structure analysis and operationalized by different complexity models, and the average score of a solution strategy as a measure for its empirical complexity. At first, solution strategies are considered and compared to each other. Then the method is extended to complete modelling tasks where both theoretical and empirical difficulties of the tasks' solution strategies are averaged to characterize the task complexity. To compare theoretical and empirical complexity, a power regression is implemented with a calculation of its goodness of fit in terms of a pseudo- R^2 -value. This allows for a statistical evaluation of the substantially reasonable relationship that "the more difficult a solution strategy is, the lower is the respective score". Furthermore, this supports a substantial conclusion with regard to the influence of parallelism of thought operations on the complexity of a solution strategy.

Complexity of Solution Strategies

A comparison of the theoretical and empirical complexity depending on the respective complexity model (see [Table 1](#)) shows that the factorial model leads to the best results concerning the goodness of fit of the regression model ([Figure 5](#)). This result speaks to the suitability of the factorial model for the characterization of the theoretical complexity of solution strategies. The factorial model weights parallel thought operations which leads to the conclusion that parallelism has an effect on the complexity of a solution strategy. The maximum model with a pseudo- R^2 -value of only 0.15 turns out to be inadequate. The addition model with a pseudo- R^2 -value of 0.69 however, proves to be adequate. The norm and quadratic model show weak pseudo- R^2 -values of 0.52, which suggests only a limited suitability.

These results suggest that parallelism of thought operations has an influence on the complexity of the solution strategy, although the kind of weighting used in the analysis of these solution strategies seems to be crucial. This is also underlined by the fact that the application of the addition model can lead to better results than most of the weighting models (maximum, norm and quadratic model) which nevertheless also indicates that a pure addition of thought operations might also be a reasonable analysis strategy.

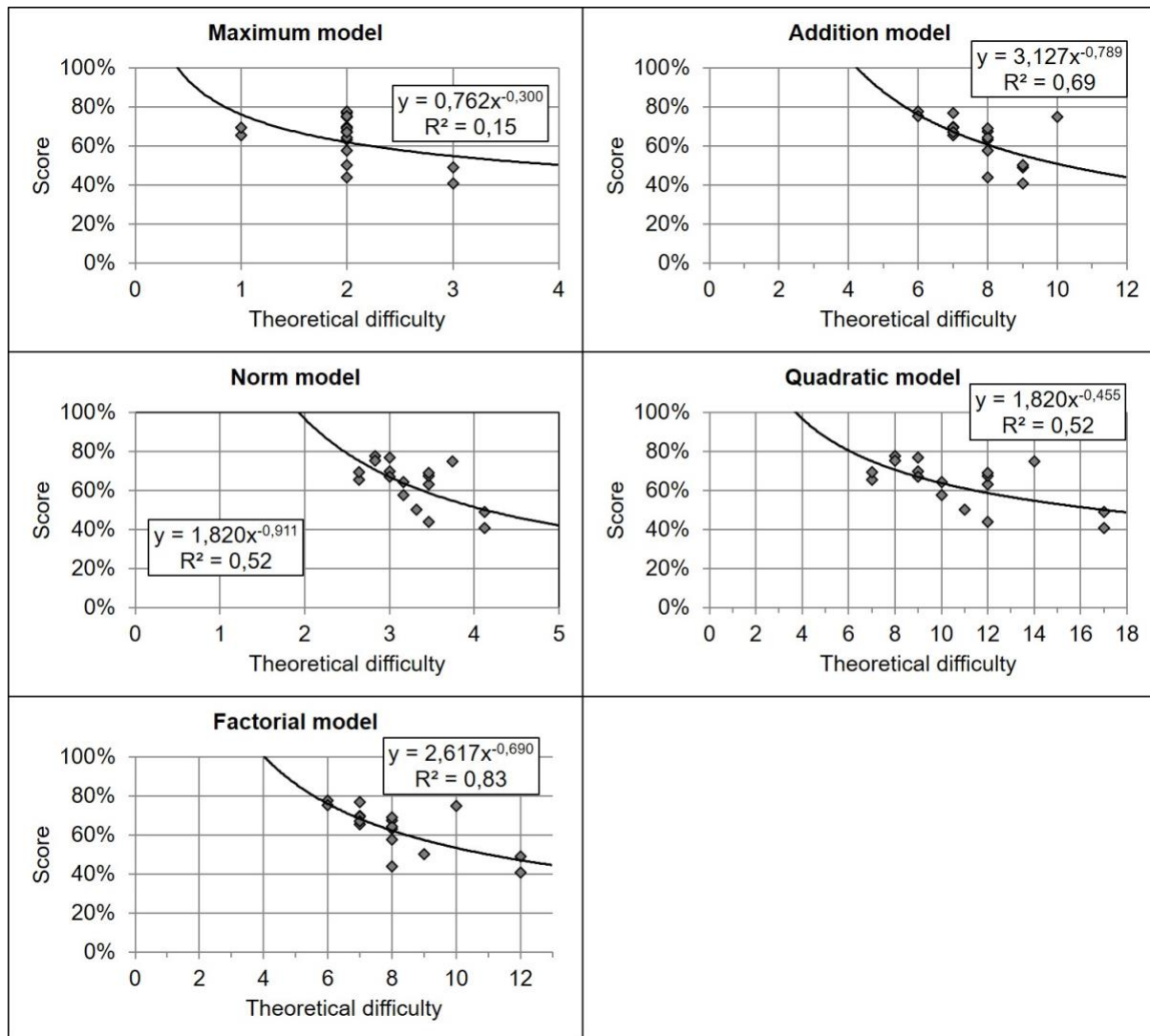


Figure 5. Comparison of the average score as a measure for the empirical complexity and theoretical complexity of solution strategies by a power regression and its goodness of fit in terms of a pseudo- R^2 -value

The main question this paper asks is whether the thought structure analysis together with its underlying cognitive and structural assumptions, leads to a reasonable and suitable characterization of the complexity of solution strategies of modelling tasks. The results presented above suggest that the factorial model is a suitable analytical tool to analyze the complexities of solution strategies when solving modelling tasks. The factorial model capitalizes on, and considers, the complexities associated with parallel thought operations, which aligns well with Sweller's cognitive load theory (Sweller, 1988). The good performance of the factorial model confirms that parallelism of thought operations has an influence on the complexity of solution strategies.

Nevertheless, it should also be emphasized that that addition model, as a non-weighting model leads to statistically good results. This suggests that an over-reliance of weighted complexity models

and the influence of parallelism of thought operations on complexity cannot be assumed. Furthermore, the statistically good results of the addition model also support the so far intuitive procedure of many mathematics teachers' assessment of mathematics tasks. A common practice among teachers' assessment of learners' solution strategies is to identify reasonable intermediate steps in a solution which are worthwhile scoring. Assuming that the assessment of solutions to a mathematics task is strongly determined by the complexity of the task, the results of the addition model supports the so far intuitive assessment practice, and we suggest that this practice can also be applied to modelling tasks.

Baird, Hopfenbeck, Newton, Stobart and Steen-Utheim (2014, p. 21) state that "assessments define what counts as valuable learning and assign credit accordingly". It is also supported by Niss' proverb that "what you assess is what you get" (Niss, 1993).

CONCLUSION

Assessment is of great importance in prescribing and guiding what happens in a mathematics class. Notwithstanding that assessment is a great challenge for many teachers in the context of modelling tasks, this study provides a promising frame of reference for further research in this important domain of assessment and modelling.

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