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RELATIONSHIP BETWEEN MATHEMATICAL LITERACY AND OPPORTUNITY TO LEARN WITH DIFFERENT TYPES OF MATHEMATICAL TASKS

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Abstract

We investigated how the opportunity to learn (OTL) with different types of mathematics tasks are related to mathematical literacy and the role of perceived control in the relationship between OTL and mathematical literacy. The structural equation modeling was applied to the data of 1,649 Korean students from the PISA 2012 database. OTL with the four different types of tasks – algebraic word problems, procedural tasks, pure mathematics reasoning, and applied mathematics reasoning – were measured via student survey on how often they have encountered each type of task in their mathematics lessons and tests. The results showed that OTL with the procedural tasks was likely to increase mathematical literacy directly and indirectly through internal perceived control. Engaging in the applied reasoning tasks is positively related to external perceived control, but negatively to mathematical literacy.

Keywords: Opportunity to Learn, Mathematical Tasks, Mathematical Literacy, Perceived Control, PISA 2012

Abstrak

Kami menyelidiki tentang bagaimana Kesempatan Belajar (KB) siswa dengan berbagai jenis tugas matematika yang terkait dengan literasi matematika dan peran *perceived control* dalam hubungan antara KB dan literasi matematika. Pemodelan persamaan struktural diterapkan pada data 1.649 siswa Korea dari database PISA 2012. KB dengan empat jenis tugas, yaitu soal cerita aljabar, tugas prosedural, penalaran matematika, dan penalaran matematika terapan, yang diukur melalui survei siswa tentang seberapa sering mereka menjumpai setiap jenis tugas dalam pelajaran dan tes matematika mereka. Hasil penelitian menunjukkan bahwa KB dengan tugas prosedural cenderung meningkatkan literasi matematika secara langsung dan tidak langsung melalui *perceived control* internal. Keterlibatannya dalam tugas penalaran yang diterapkan, bernilai positif terhadap *perceived control* eksternal, namun bernilai negatif terhadap literasi matematika.

Kata Kunci: Kesempatan Belajar, Soal Matematika, Literasi Matematika, *Perceived Control*, PISA 2012

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Since Carroll (1963) introduced the concept of opportunity to learn (OTL), OTL has been conceptualized as the inputs and processes that are needed to produce student achievement of intended outcomes (Elliott & Bartlett, 2016). Building on the early conceptualization of OTL as allocation of learning time (Carroll, 1963; Cogan & Schmidt, 2015), studies on mathematical practices and OTL are concerned with the processes through which individuals come to know mathematics content (Barnard-Brak et al., 2018). A contemporary definition of OTL is comprised of factors that have a significant influence on teachers' instructional practices and students' learning (Stevens & Grymes, 1993); these factors include content coverage and emphasis. Content coverage refers to which concepts and cognitive skills of curricula are covered during classroom learning, whereas content emphasis is related to activities and tasks that engage students (Stevens & Grymes, 1993).

In this study, OTL in mathematics classrooms is conceptualized as mathematical tasks that allow learners to have actual experiences with mathematics, which focuses on OTL as content emphasis in learning tasks (Schoenfeld, 1992). Learners' cognitive processes are shaped through the experiences with learning tasks. Further, by engaging in the mathematical tasks, students develop their understanding of what it means to *do mathematics* (Schoenfeld, 1994). By tackling various types of mathematical tasks, students can do mathematics and construct an epistemological understanding of what doing so means. Therefore, we conceptualize OTL as cognitive processes that learners engage in while doing mathematics through engaging in different types of tasks.

OTL, which is conceptualized around mathematical tasks, is related to students' learning outcomes, according to the framework suggested by Stein et al. (1996). However, a large body of literature based on this framework has focused on teachers' instructional practices, such as teachers' implementation of mathematical tasks that require high cognitive demands. There are few previous research studies that exist on the relationship between students' OTL and their learning outcomes, and these studies' conceptualization of OTL did not include mathematical tasks used in the mathematics instruction. For example, OTL in the study of Ottmar et al. (2014) consisted of two dimensions: instructional quality (teachers' efforts to promote reasoning and understanding of concepts via teacher-student interaction) and exposure to mathematics instruction (how long students were exposed to mathematics instruction). To bridge this research gap, we question whether there is a relationship between OTL (students' exposure to different types of mathematical tasks) and mathematics achievement.

Due to the complex nature of learning environments (Berliner, 2002; Jacobson et al., 2019) we hypothesize that the relationship between our conceptualization of OTL (frequency and type of mathematics tasks) and mathematics achievement is not only a direct, but also an indirect relationship through other factors. According to the framework of mathematical instructional tasks (Stein et al., 1996), the mathematical tasks that are set up by the teacher interact with and shape students' dispositions, including attitudes, beliefs, and motivation. This interaction between mathematical tasks and learning disposition in turn, influences students' cognitive processes and learning behaviors (Henningsen & Stein, 1997). Finally, the overall processes that involve mathematical tasks and students' perception are reflected in the students' learning behaviors and outcomes. For this reason, we consider students' perceived control as a meditating factor between OTL and achievement. Students who believe that academic outcomes are under their own control are predicted to be more actively engaged in mathematical tasks and earn better academic outcomes. This interaction between learning tasks and students' disposition (specifically, perceived control in this study) has not been found in previous literature.

This study is a secondary analysis that uses the database from the Program for International Student Assessment (PISA) 2012. Using extensive data from the PISA 2012 database, we investigated the relationship between OTL, mathematical literacy, and perceived control using structural equation modeling (SEM) approach. In this study, we focused on one educational context, South Korea, rather than examining different contexts of multiple countries. Before making international comparisons, an exploratory study to understand the phenomenon in a single context would be required for meaningful conclusions. Moreover, the rationale for selecting South Korea is that it is one of the high achieving countries, and that it has not been fully investigated in terms of OTL (Son, 2012). Furthermore, the OECD working paper (Schmidt, Zoido, & Cogan, 2014) showed that in each country, there is linear or quadratic relationship between exposure to different types of mathematical tasks and mathematical literacy. Since we focus on Korean student data and include perceived control in SEM analysis, our study can provide a broader picture of the relationship among different types of mathematics tasks, perceived control, and mathematical literacy.

The purpose of this study is to explore the relationship between OTL – a combination of exposure and types of mathematical tasks – and mathematical literacy measured in the PISA 2012. The nature of the relationship between OTL and mathematical literacy is not direct, and rather, the relationship is mediated by perceived control. OTL is hypothesized to be related to mathematical literacy via students' perceived control.

Opportunity to Learn

Within an educational context, OTL refers to the inputs and processes that are provided to students for intended learning outcomes (Elliott & Bartlett, 2016). One of the first conceptualizations of OTL focused on sufficient time and adequate instruction to learn (e.g., Carroll, 1963; Schmidt, 1992). With growing interest in the concept of OTL in relation to the demand for curricular validity, the concept of OTL has been expanded to accommodate a multi-dimensional construct that encompasses both the quality of instruction and its alignment with the assessment of learning outcomes (Abedi & Herman, 2010). Specifically, Stevens (1996) proposed a comprehensive conceptual framework of OTL that includes four elements: content coverage, content exposure, content emphasis, and quality of instructional delivery. As such, OTL, as a comprehensive and multi-dimensional concept, offers a basis for investigating students' learning in the mathematics classroom (e.g., Abedi & Herman, 2010).

When considering how these different dimensions of OTL are realized in the mathematics classroom, it is clear that mathematical tasks serve as a critical learning space that provide students with experiences of mathematical practice. In other words, mathematical tasks that comprise different dimensions of OTL (e.g., content coverage, content exposure, content emphasis, and quality of instructional delivery) interact with and, in turn, shape students' learning processes, both cognitive and non-cognitive. For example, in the studies of Watson (2003) and Törnroos (2005), class tasks, in addition to the curriculum and the textbook, were identified as one of the critical aspects of OTL.

In the PISA 2012, OTL is conceptualized as a constellation of three constructs that describe classroom learning environments: (1) measurement of content, (2) teaching practices, and (3) teaching quality (OECD, 2013). According to Schmidt et al. (2014), OTL in the PISA 2012 refers to the content students learn, as well as the cohesiveness that exists between what is taught and what they actually learn. Also, students' experiences with mathematical content are shaped by instructional practices, including student-centered instruction and lectures. Students' OTL is characterized by the factors underlying the quality of instructional practices, such as classroom organization, emotional support, and cognitive activation (OECD, 2013).

In the frameworks that are used in previous studies and PISA 2012, the concept of OTL includes specific content that is covered in mathematics classrooms, as well as mathematical tasks that deliver mathematics content. On one hand, the commonality of these frameworks is that mathematical tasks are an important factor of mathematics learning, and that teachers can affect students' cognitive and motivational processes of learning by designing these tasks. On the other hand, we also recognize differences among the frameworks that conceptualized OTL. One difference between the PISA 2012 framework and other literature on OTL is that in the PISA, OTL is operationalized as students' judgment on whether and how often they have encountered different mathematical tasks. This operationalization for measurement is partly limited in covering depth of teaching or quality of instructional delivery variables (E.g., Stevens, 1993), which is also recognized in the PISA 2012 framework (OECD, 2013, p.187). As such, we do recognize the multifaceted characteristic of OTL, but also acknowledge that large-scale assessment would not be enough to fully understand OTL that students experience in mathematics classrooms as reported in the PISA 2012 framework. In this study, we assume the operationalization of OTL in the PISA 2012, student-reported frequency of being exposed to different types of mathematical tasks.

Mathematical Tasks

Among the multiple aspects of OTL, we highlight students' exposure to different types of mathematical tasks in lessons and tests, as the tasks themselves reflect what content the students learn and what doing mathematics entails (Stein et al., 1996). In other words, mathematical tasks are essential tools for ensuring that students can understand mathematical concepts more fully, as well as to develop cognitive processes of mathematical reasoning via their experience with the tasks (Martin & Gourley-Delaney, 2014).

With regard to the cognitive processes of learning, the students' experience of mathematics depends on the level of cognitive demands, how the tasks are presented, and how the tasks are implemented. Adopting the conceptual framework regarding the relationship between variables that are related to tasks and students' learning outcomes (Stein et al., 1996), many studies have shown that cognitive demands of mathematical tasks can change as they are implemented (e.g., Boston & Smith, 2009; Henningsen & Stein, 1997). When students engage in mathematics, their reasoning differs according to what type of mathematical tasks are being offered (see *Potential of the Task* in Boston & Smith, 2009). Mathematical thinking processes that students employ are closely related to the mathematical tasks that are embedded in the learning context (Henningsen & Stein, 1997). Certainly, the elements and characteristics of mathematical tasks require students to engage in different cognitive processes. Hanna and Jahnke (2007) provided a good example by comparing activities that involve either pure mathematics or a real-life situation. Proving statements is a combination of two processes: "(1) finding the 'right premises' and (2) devising the chain of deductive steps leading from the premises to the statement" (p. 149). Mulnix (2012) labeled these as the process of *searching for reasons* (e.g., abduction/induction) and the process of *giving reasons* (e.g., deduction). According to Hanna and Jahnke (2007), the process of giving reasons is more emphasized in tasks that involve pure mathematical reasoning, which is why the process of searching for reasons has usually been downplayed. In contrast, mathematical tasks with real situations require setting up the premise first (searching for reasons), which is followed by the process of building logical connectedness (giving reasons).

Previous studies have scrutinized mathematical tasks set up by teachers and teachers' actual implementation of the tasks based on the framework developed by Stein et al.'s (1996) (e.g., Arbaugh & Brown, 2005; Boston & Smith, 2009). However, few studies have been conducted to investigate the link between mathematical tasks and students' learning outcomes, particularly measured by large-scale assessments. This is possibly because of the assumption that large-scale assessments are designed to evaluate students' content knowledge, not their mathematical practices (Lane, 2004). However, we argue that students formulate and utilize epistemic and cognitive resources to reason through OTL with mathematical tasks (Hammer, 2000), and students utilize some of those resources to solve problems in assessments (Bailin & Siegler, 2003; Hwang et al., 2020). The common cognitive resources used while engaging in mathematics tasks and assessment settings can help us to understand how students' OTL is connected to achievement scores in large-scale assessments.

The relationships between OTL and achievement can also be influenced differ by what mathematics tasks are involved in students' OTL. Individual differences in mathematics learning can be understood as interaction between features of tasks and students' inputs (i.e., cognitive resources, affectivity; Bornemann et al., 2010; Muis et al., 2015). As discussed, students' mathematical reasoning, as one of the critical components of doing mathematics, differs by what type of mathematical tasks are offered to them. The emotional components, such as task valuing and perceived control (Muis et al., 2015), can motivate them to either continue reasoning or terminate the reasoning process (McLeod, 1992). Particularly, students' perceived control – "the tendency of people to perceive that outcomes in a particular arena were either within or outside of their control" (McNabb, 2003, p. 418) – influences students' approaches to solving mathematical tasks. For example, students are likely to engage more actively when they believe that the outcomes from engagement in tasks are under their control (Hrbáčková et al., 2012).

Perceived Control

Control beliefs refer to an overall set of beliefs about how effective one's process of producing expected outcomes can be (Skinner et al., 1998). In academic settings, perceived control is understood as a critical psychological disposition that affect students' behavior, emotion, and achievement (d'Ailly, 2003; Schunk, 1984; Murayama et al., 2013). According to the previous frameworks of perceived control (e.g., Skinner et al., 1998; Rotter, 1966; Rotter & Mulry, 1965), perceived control over learning is constituted of two types of beliefs: strategy beliefs (what it takes to do well) and capacity beliefs (whether I believe I have the strategies; Skinner et al., 1998). According to Rotter (1966), people differ in their beliefs whether outcomes occur independently of how one behaves (external control) or are highly contingent on one's behavior (internal control). The construct of locus of control assume that internal and external causes are inversely related to each other and thus, can be assessed as a single, bipolar dimension (Skinner et al., 1990). Though perceived control has been shown to be an important indicator of students' motivation in learning (Patrick et al., 1993), previous studies on perceived control rarely examined it in the relation to the success in academic tasks (Skinner et al., 1990; Lipnevich et al., 2016).

In PISA 2012, the conceptualizations of perceived control and other self-perceptions are based on the *planned behavior theory* of Ajzen (2002). According to Ajzen (2002), perceived control belief is conceptualized as a person's belief about the ease or difficulty of performing a behavior and this belief forms a behavioral intention that directly increases the likelihood of a desired behavior. In this study, we viewed locus of control as having two types and identified student survey items that ask about their locus of control over mathematics learning and categorized them into internal and external perceived control. It is assumed that a learner's strong internal perceived control does not necessarily lead to weak external perceived control, and furthermore, they are qualitatively different with various sources of beliefs.

Mathematical Literacy in the PISA

The relationships between OTL and achievement can differ by how achievement is defined, and with what measure it is assessed. In this study, achievement scores represent students' mathematical literacy measured with the PISA 2012. According to the PISA 2015 framework, mathematical literacy "explains the processes content knowledge, and contexts reflected in the assessment's mathematics problems", and this shows how students perform in mathematics (OECD, 2017). The construct of mathematical literacy describes competency of individuals to reason mathematically and use math concepts, procedures, facts, and tools to describe, explain, and predict phenomena (OECD, 2017). This conceptualization of mathematical literacy supports the importance of students' engagement in pure mathematics tasks (reason mathematically) and their exploration in the abstract world of mathematics (use math concepts, procedures, facts, and tools; OECD, 2017).

When contemplating PISA's definition of mathematical literacy, it also emphasizes the capacities to formulate problem situations, employ mathematical problems, and interpret mathematics results in various contexts. In other words, rich experiences of real-world tasks in math classrooms are essential in developing these capacities. Accordingly, having experiences of doing mathematics in real world contexts (personal, societal, occupational, and scientific situations) contributes to the development of mathematical literacy.

Hypothesized Model

According to the literature review, we suggest the hypothesized model in Figure 1 representing the relationships among OTL by different types of tasks, perceived control, and mathematical literacy. This model includes a latent variable for each type of mathematical tasks (word problems, procedural tasks, pure mathematics reasoning, and applied mathematics reasoning tasks. In addition, there are two latent variables for internal and external perceived control in the model. By evaluating the appropriateness of the hypothesized model, we aimed to answer the following questions: (1) what are the relationships between opportunities to learn with different types of tasks and perceived control? (2) what are the relationships between opportunities to learn with different types of tasks and mathematical literacy measured in the PISA 2012?

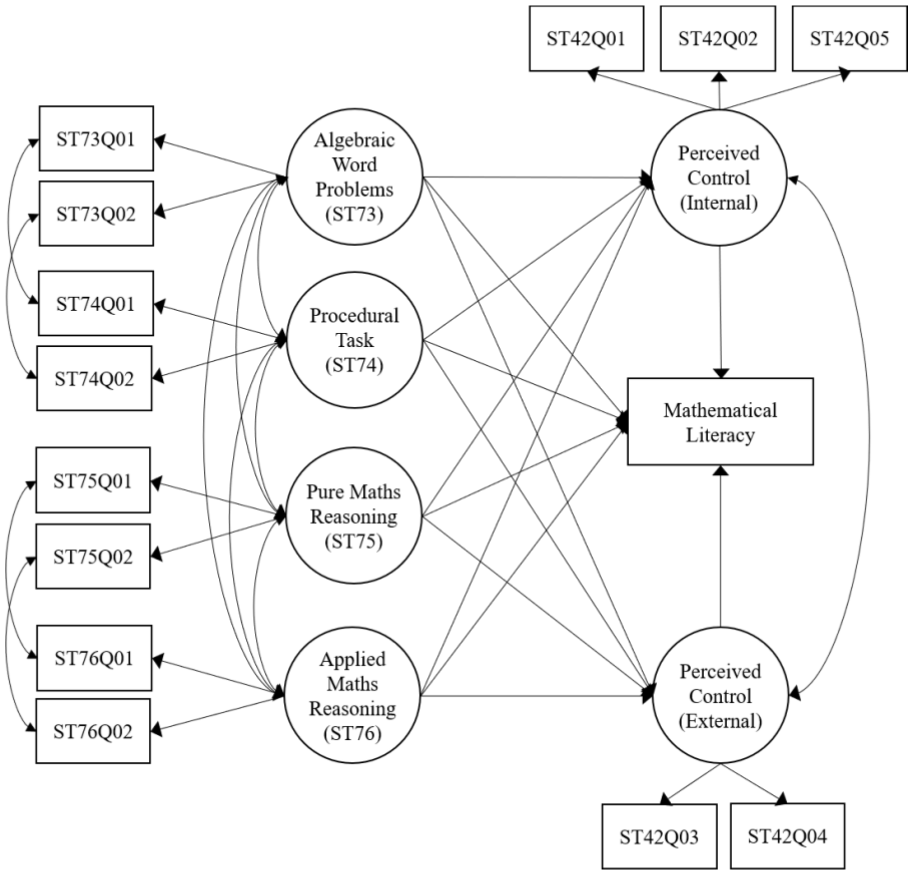


Figure 1. Full Model of the Relationship among OTL, Perceived Control, and Mathematical Literacy

It should be noted that each of the latent variables of OTL and EPC was estimated using two indicators due to the structure of the PISA 2012 data. It is commonly recommended to have more than two indicators per latent variable. However, some researchers argued that one or two indicators could be sufficient (Hayduk & Littvay, 2012). Furthermore, the analysis did not yield any errors that are very likely to happen with two indicators (e.g., negative residual variances known as Haywood cases), and the number of indicators showed little effect of bias (Little et al., 1999). For those reasons, the estimated model is valid to interpret the relationship between OTL and mathematical literacy.

Thus, we are interested in both direct relationship and indirect relationships through perceived control, between mathematical tasks and mathematical literacy. We attempted to test the positive relationships between OTL with different tasks and mathematical literacy by examining the hypothesized model with the structural equation modeling. When students learn mathematics through OTL with mathematical tasks, the positive relationships between the tasks and mathematical literacy are somewhat expected. Particularly, we expected that students' opportunity to engage in applied mathematics reasoning tasks would be more strongly related to mathematical literacy based on the definition of mathematical literacy given by the PISA 2015.

METHOD

Participants

We utilized the PISA 2012 international database, which is open to the public. The rationale to use this database instead of the PISA 2015 or 2018 was that the focus subjects of these recent PISA studies were not mathematics. The variables included to address our research questions were available only in the PISA 2012. Among 5,033 Korean students in the original PISA 2012 database we collected responses of 1,649 Korean students who participated in both student questionnaire and mathematical literacy assessment. The PISA 2012 student context questionnaires in a rotation design, which consisted of the 'common' question (answered by all students) and 'rotated' questions (answered by two thirds of the student sample; OECD, 2014, p. 59). Because all of the survey items used in this research (ST43, questions asking 'Thinking about your mathematics lessons: to what extent do you agree with the following statements?' and ST73-76, mathematical tasks) were included together in the form A, the rotation questionnaires design allowed us to observe only students taking this form, a third of the Korean students participating in the PISA 2012 (See figure 3.9 in OECD, 2014, p. 61).

Variables

Mathematical literacy. As seen in the hypothesized model, we collected all sets of plausible values representing students' mathematical literacy scores provided in the PISA 2012. Large-scale international studies such as TIMSS and PISA do not provide one value for each student's achievement in mathematics. Rather, as Foy, Brossman, and Galia (2012) argued, plausible values are provided through the process called "conditioning" with all background variables, for which relationships between background variables and mathematics achievement can serve as a satisfactory explanation.

Table 1. Weighted Means and Standard Deviations of Mathematical Literacy

Plausible Value	PV1	PV2	PV3	PV4	PV5
Weighted Mean	553.57	553.53	555.08	553.56	553.44
Standard Deviation	99.61	100.22	99.53	100.52	101.30

Furthermore, we highlighted that "[p]lausible values are not test scores and should not be treated as such" (OECD, 2014, p. 147) and the plausible values should be analyzed in a correct way. According to von Davier, Gonzalez, and Mislevy (2009), averaging plausible values themselves to have one value representing students' mathematical literacy could lead to biased estimates. Chaney et al. (2001) suggested conducting separate analysis with each set of plausible values and average the estimated parameters. We also applied some formulas that Chaney and his colleagues provided to compute the standard errors for calculated estimates. Lastly, mathematical literacy was standardized in the SEM analysis. Table 1 shows the weighted mean and the standard deviation of each set of the plausible values.

Perceived control. We collected students' responses to the question, given the code, ST43, asking students' degrees of agreements to six statements in Table 2. For data analysis, we assigned "4" to students' strong agreement to each statement, "3" to moderate agreement, "2" to moderate disagreement, and "1" to strong disagreement. Though the way of assigning numbers to students' responses is different from the way used in the PISA 2012, our method allows us to interpret that higher numbers of students' responses indicate stronger agreement to the statements about perceived control.

After selecting the data of the question ST43, we categorized the six statements into the two: internal (IPC) and external perceived control (EPC) based on the discussion to build the hypothesized model. Internal perceived control was measured through the three statements – ST43Q01, ST43Q02, and ST43Q05. Other two statements – ST43Q03 and ST43Q04 – were used to measure external perceived control. We excluded the statement, ST43Q06 that asked about test preparation because it is neither internal nor external perceived control. This statement can imply that students' performance on mathematics exams is irrelevant to their efforts, but the statement itself does not allows us to identify the statement as either internal or external perceived control.

As seen in Table 2, more than 85% of Korean students agreed with the three statements ST43Q01, ST43Q02, and ST43Q05, which were used to measure internal perceived control. Simultaneously, most Korean students disagreed with the other statements about external perceived control and test preparation. When ST43Q03 and ST43Q04 were compared, it was interesting that more students strongly agreed that their success/failure is attributed to their teachers.

Table 2. The Number of Students and Weighted Percentage by Response to the Six Statements

Code	**Question:** Thinking about your mathematics lessons: to what extent do you agree with the following statements? Statement	Frequency			
		Strongly Agree (4)	Agree (3)	Disagree (2)	Strongly Disagree (1)
ST43Q01	If I put in enough effort I can succeed in mathematics.	503 (30.1%)	945 (57.7%)	166 (10.1%)	35 (2.1%)
ST43Q02	Whether or not I do well in mathematics is completely up to me.	612 (37.0%)	912 (55.5%)	93 (5.6%)	32 (1.9%)
ST43Q03	Family demands or other problems prevent me from putting a lot of time into my mathematics work.	66 (3.9%)	311 (18.8%)	940 (57.2%)	332 (20.1%)
ST43Q04	If I had different teachers, I would try harder in mathematics.	116 (7.1%)	333 (20.0%)	912 (55.3%)	288 (17.6%)
ST43Q05	If I wanted to, I could do well in mathematics.	482 (29.0%)	951 (57.9%)	172 (10.5%)	44 (2.6%)
ST43Q06	I do badly in mathematics whether or not I study for my exams.	108 (6.6%)	425 (26.0%)	793 (48.2%)	323 (19.2%)

Opportunity to Learn. Students' OTL data were collected with the responses to four questions (ST73, ST74, ST75, and ST76). Each question included two sub-questions showing different types of tasks: "how often have you encountered these types of problems in your mathematics lessons (ST[73–76]Q01)?; and "in the tests you have taken at school (ST[73–76]Q02)?" We highlight that the four questions focused on students' perception of *how often* they encountered OTL with the tasks, among other dimensions of OTL (Stevens & Grymes, 1993). Table 3 shows the detailed questions for OTLs labeled with *algebraic word problem* (WP; ST73) and *procedural tasks* (PT; ST74) in the PISA 2012. Table 4 also shows the other two questions used to identify OTL with *pure mathematics reasoning* (PMR) and *applied mathematics reasoning* (AMR).

Table 3. OTL Questions for Algebraic Word Problem and Procedural Task (OECD, n.d.)

Algebraic Word Problem (WP; ST73)	**Question.** In the box is a series of problems. Each requires you to understand a problem written in text and perform the appropriate calculations. Usually the problem talks about practical situations, but the numbers and people and places mentioned are made up. All the information you need is given. Here are two examples: **Example 1)** <Ann> is two years older than <Betty> and <Betty> is four times as old as <Sam>. When <Betty> is 30, how old is <Sam>? **Example 2)** Mr. <Smith> bought a television and a bed. The television cost <$625> but he got a 10% discount. The bed cost <$200>. He paid <$20> for delivery. How much money did Mr. <Smith> spend?				
	# of Students	Frequently (4)	Sometimes (3)	Rarely (2)	Never (1)
	ST73Q01	545 (33.1%)	821 (49.8%)	215 (13.1%)	68 (4.0%)
	ST73Q02	336 (20.5%)	792 (48.2%)	410 (24.7%)	111 (6.6%)

Procedural Tasks (PT; ST74)	**Question.** Below are examples of another set of mathematical skills. **Example 1)** Solve 2x + 3 = 7. **Example 2)** Find the volume of a box with sides 3m, 4m and 5m.				
	# of Students	Frequently (4)	Sometimes (3)	Rarely (2)	Never (1)
	ST74Q01	968 (58.9%)	540 (32.6%)	109 (6.6%)	32 (1.8%)
	ST74Q02	746 (45.5%)	623 (37.8%)	217 (13.1%)	63 (3.7%)

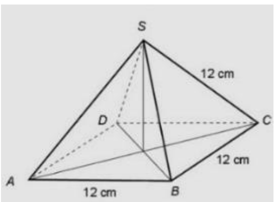
Note. All percentages were weighted.

An interesting finding from Tables 3 and 4 is that more than 70% of students answered that they encountered each of WP, PT, and PMR frequently in their mathematics lessons and tests. However, approximately a half of students reported that they encountered AMR frequently or sometimes. Particularly, 91.4% of students encountered PT at least sometimes, whereas 44.2% of students saw AMR tasks at most rarely in lessons. This indicates that there were substantial gaps in the frequencies of OTL with different tasks that were offered to Korean students; specifically, limited OTL with AMR that requires students to make sense of real problem situations and interpret/explain the solutions. This is a unique mathematical process that AMR offers, while other task types do not.

Table 4. OTL Questions for Pure Mathematics Reasoning and Applied Mathematics Reasoning (OECD, n.d.)

Pure Mathematics Reasoning (PMR; ST75)

Question. In the next type of problem, you have to use mathematical knowledge and draw conclusions. There is no practical application provided. Here are two examples.

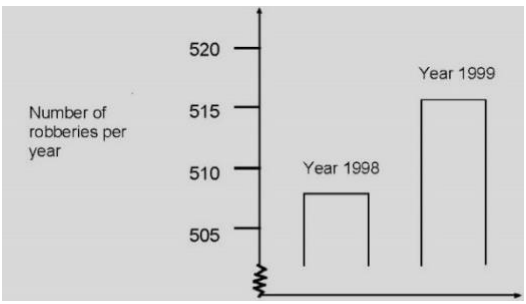


Example 1) Here you need to use geometrical theorems: Determine the height of the pyramid.
Example 2) If *n* is any number: can $(n+1)^2$ be a prime number?

# of Students	Frequently (4)	Sometimes (3)	Rarely (2)	Never (1)
ST75Q01	560 (34.0%)	722 (44.1%)	281 (16.8%)	86 (5.0%)
ST75Q02	511 (31.0%)	719 (44.0%)	308 (18.6%)	111 (6.5%)

Applied Mathematics Reasoning (AMR; ST76)

Question. In this type of problem, you have to apply suitable mathematical knowledge to find a useful answer to a problem that arises in everyday life or work. The data and information are about real situations. Here are two examples.
Example 1) A TV reporter says "This graph shows that there is a huge increase in the number of robberies from 1998 to 1999."



Do you consider the reporter's statement to be a reasonable interpretation of the graph? Give an explanation to support your answer.
Example 2) For years the relationship between a person's recommended maximum heart rate and the person's age was described by the following formula:

Recommended maximum heart rate = 220 – age

	Recent research showed that this formula should be modified slightly. The new formula is as follows: *Recommended maximum heart rate = 208 – (0.7 × age)* From which age onwards does the recommended maximum heart rate increase as a result of the introduction of the new formula? Show your work.			
# of Students	Frequently (4)	Sometimes (3)	Rarely (2)	Never (1)
ST76Q01	203 (12.4%)	717 (43.6%)	564 (34.1%)	165 (9.8%)
ST76Q02	187 (11.3%)	631 (38.6%)	618 (37.2%)	218 (12.9%)

Note. All percentages were weighted.

Data Analysis

Using the variables described above, we applied the structural equation modeling (SEM) to evaluate the hypothesized model in Figure 1. The SEM approach was utilized with the R package *lavaan.survey* (Obserski, 2016) and maximum likelihood estimation that considered all variables as continuous. The strength of this R package was that the complex PISA 2012 hierarchical design could be fully considered in the SEM analysis using students' weights and balanced repeated replications (BRR). First, because our research interests were solely at the student level, the data analysis required to use student weights in data analysis (Asparouhov & Muthen, 2006). The PISA 2012 provided "final trimmed nonresponse adjusted student weight," which was calculated with the consideration of stratified sampling design. Thus, statistical results such as descriptive statistics and SEM results were weighted. Second, weighting was not enough to make unbiased decisions when multilevel sampling was applied. Particularly, "the variance estimator can be unstable" relying on the sample design (OECD, 2017, p. 123). To resolve this issue, it was recommended to use BRR to estimate sampling variances (OECD, 2017). In this research, we employed Fay's method of BRR by using variables "final student replicate BRR-Fay weights" in the databases.

Considering the complexity of the analysis using the five sets of plausible values and Fay's method of BRR, we applied the three steps of the SEM approach suggested by Byrne (1998): model specification, model assessment, and model respecification. First, as discussed in the previous section, the hypothesized model in Figure 1 was already constructed based on the relevant literatures. Second, the followings were evaluated for the next stage of respecification: the overall model fits, the suitability of parameter estimates, and the statistical significance of parameter estimates. It was checked that the outputs included some error messages like negative variances, correlations greater than 1, and non-positive definite covariance matrices, which all are unreasonable (Bryne, 1998). Furthermore, non-significant parameters could slightly contribute to the power of the model to explain the phenomenon. Thus, we considered the parameters with $p < 0.1$ because we had less concern of Type I error. For the overall model evaluation, all model fits were comprehensively evaluated using criteria summarized by Schreiber, Nora, Stage, Barlow, and King (2006, p. 330) – the comparative fit index (CFI), the Tucker-

Lewis fit index (TLI), the standardized root mean square residual (SRMR), and the root mean square error of approximation (RMSEA).

On one hand, we removed some non-significant indicators to respecify the hypothesis model after model assessments. On the other hand, we included correlated residuals having large modification indices representing expected changes in model fits. However, we should have theoretical rationales in addition to the statistical evidence to add correlated residuals (Kline, 2011). Accordingly, we considered four pairs of correlated residuals: (1) ST73Q1 and ST74Q1, (2) ST73Q2 and ST74Q2, (3) ST75Q1 and ST76Q1, and (4) ST75Q2 and ST76Q2. These pairs showed statistical evidence of large modification indices. Also, it is noticeable that residuals of the questions about WP and PT, and PMR and AMR were correlated, which could be due to the cognitive demands of mathematics tasks (Boston & Smith, 2009).

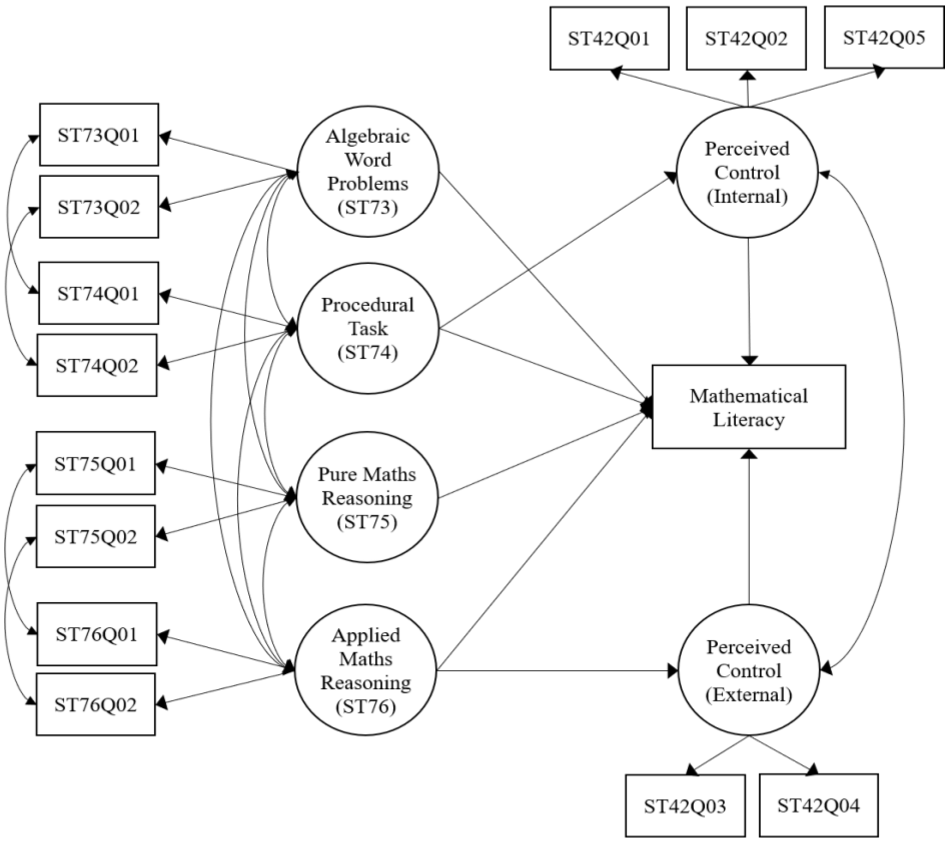


Figure 2. Nested Model Respecified from the Full Model

After the respecification, we compared the full model (see Figure 1) and the nested model (see Figure 2). Using χ^2 tests, AIC, BIC, and sample-size adjusted BIC, the comparison could answer whether there was a significant difference in the goodness of fit between the nested and full models, though the nested model had a smaller number of parameters. If we retain the null hypothesis that there is no significant difference between the models, we prefer the nested model to the full model, because the nest model had a similar goodness of fit with less parameters. Then, the SEM results of the relationships among OTLs, perceived control, and mathematical literacy would be based on the regression weights in the selected model.

We highlight that estimation methods in SEM (e.g., maximum likelihood in this study) require a normality assumption of endogenous variables. However, it is known that parameter estimates are robust against violation of normality assumption while Type I error rate of hypothesis tests on individual parameters are likely to be inflated. Furthermore, plausible values of mathematical literacy measured in PISA 2015 were constructed based on normal population distributions. Thus, we argued the robustness of the model.

RESULTS AND DISCUSSION

We will report the fit indices to compare the nested and full models to answer our research questions. After the discussion of model selection process, we will report the SEM results to explain the relationships among OTLs, perceived control, and mathematical literacy.

Table 5. Model Fit Indices using Five Plausible Values

				Information Index			RMSEA		
		CFI	TLI	AIC	BIC	Adjusted BIC	Point Estimate & 90% Confidence Interval	SRMR	χ^2 test *p*-value
PV1	Full	0.977	0.962	43687.9	44036.9	43830.4	0.049 (0.043 0.055)	**0.021**	0.784
	Nested	0.977	**0.966**	**43679.6**	**43996.3**	**43808.9**	**0.046 (0.040 0.052)**	0.022	
PV2	Full	0.977	0.961	43715.9	44064.9	43858.4	0.049 (0.043 0.055)	**0.022**	0.785
	Nested	0.977	**0.965**	**43707.5**	**44024.3**	**43836.8**	**0.046 (0.041 0.052)**	0.023	
PV3	Full	0.976	0.960	43678.0	44027.0	43820.5	0.050 (0.044 0.056)	0.023	0.780
	Nested	**0.977**	**0.964**	**43669.7**	**43986.4**	**43799.0**	**0.047 (0.041 0.053)**	0.023	
PV4	Full	0.978	0.962	43702.3	44051.3	43844.8	0.048 (0.042 0.054)	**0.022**	0.784
	Nested	0.978	**0.966**	**43693.9**	**44010.7**	**43823.3**	**0.046 (0.040 0.051)**	0.023	
PV5	Full	0.977	0.961	43702.6	44051.6	43845.1	0.049 (0.043 0.055)	**0.022**	0.787
	Nested	0.977	**0.966**	**43694.2**	**44011.0**	**43823.5**	**0.046 (0.040 0.052)**	0.023	

Note. Bold numbers indicate the better model between the nested and full models.

Model Comparison

The model fit indices of the full and nested models were estimated across all sets of plausible values of mathematical literacy. Based on the recent criteria (CFI ≥ 0.95, TLI ≥ 0.95, RMSEA < 0.06 with confidence interval, and SRMR ≤ 0.08; Schreiber et al., 2006), all indices of both models were acceptable. When we compared the full and nested models, the χ^2-test results indicated that there were no significant differences in the goodness of fit between the two models as seen in the last column of Table 5. Additionally, most indices – CFI, TLI, RMSEA – showed that the nested model had slightly better fit indices with the smaller number of the parameters. Less values of information indices (AIC, BIC, and adjusted BIC) indicated a better model, which led to the same conclusion. Thus, we selected the nest model in Figure 2, which was a more parsimonious model with the similar goodness of fit.

SEM Results

Table 6 reports the estimated measurement model in the standardized metric. Lower factor loadings indicate that the corresponding indicators were conceptually distant from the latent variables. All factor loadings except for that of ST43Q03 were greater than 0.4 with $p < 0.001$, which satisfied previously suggested recommendations (Tabachnick & Fidell, 2007). Although the factor leading of ST43Q03 was 0.387, we argue that this coefficient was acceptable. However, the factor loadings for external perceived control were similar, which means that both statements can reflect conceptually similar distance of different facets of external perceived control – teachers and family.

Table 6. Results from the Measurement Model

Observed Variable	Latent Variables	Coefficient	SE	z-value	*p*-value
ST73Q01	WP	0.675	0.026	25.502	<0.001
ST73Q02	WP	0.621	0.022	27.759	<0.001
ST74Q01	PT	0.632	0.023	27.447	<0.001
ST74Q02	PT	0.642	0.022	28.933	<0.001
ST75Q01	PMR	0.786	0.022	35.501	<0.001
ST75Q02	PMR	0.751	0.022	33.926	<0.001
ST76Q01	AMR	0.746	0.021	35.454	<0.001
ST76Q02	AMR	0.744	0.021	35.330	<0.001
ST43Q01	IPC	0.540	0.018	29.934	<0.001
ST43Q02	IPC	0.444	0.019	23.347	<0.001
ST43Q05	IPC	0.505	0.017	29.676	<0.001
ST43Q03	EPC	0.387	0.056	6.910	<0.001
ST43Q04	EPC	0.439	0.069	6.332	<0.001

The SEM results included the correlation coefficients between OTLs with different types of tasks as seen in Table 7. Overall, all OTLs were highly correlated with each other, which means that if students have more frequent OTLs with a certain type of task, they were very likely to do so with others. It should be noted that the correlation coefficient between AMR and PT was relatively low, 0.258. This correlation indicates that AMR with PT was somewhat independent compared to other pairs of OTLs.

Table 7. Correlation Coefficients between Latent Variables

	WP	PT	PMR	AMR
WP	1	0.572	0.387	0.396
PT		1	0.426	**0.258**
PMR			1	0.428
AMR				1

Note. All correlation coefficients are significant with $p < 0.01$

Figure 3 shows all SEM results based on the nested model. Table 8 reports the regression weights, which were all significant at the alpha level of 0.05. First, only the regression weight of AMR was negative (-0.174), the others were positive (0.132 for WP, 0.099 for PT, and 0.104 for PMR). The degree of this negative effects was also larger than others. IPC has much stronger relationship to mathematical literacy than EPC. It is remarkable that EPC is negatively related to mathematical literacy although it is not statistically significant at alpha 0.05. In addition, students' EPC was expected to increase by 0.091 when students had increase of 1 in their AMR. These findings about AMR indicate that encountering AMR frequently had negative influences on their mathematical literacy and positive influences on EPC simultaneously. However, OTL with PT was expected to increase mathematical literacy scores both directly and indirectly through IPC. When students increased 1 in their PT, students were expected to have increase in their IPC by 0.297. This value was remarkably large when considering that IPC was a psychological factor and that this result indicated possible impact of tasks introduced to students on their psychological perception. Because students' increase in IPC by 1 was expected to increase mathematical literacy by 0.358, the indirect effect of PK was 0.106 = 0.297×0.358 (SE = 0.016, $p < 0.001$). This value was similar with the direct effect of PT, 0.099 ($p = 0.004$).

Table 8. Results from the Structural Equation Modeling

Independent Variable	Dependent Variable	Coefficient	SE	z-value	p-value
PT	IPC	0.297	0.039	7.604	<0.001
AMR	EPC	0.091	0.043	2.120	0.034
IPC	Mathematical Literacy	0.358	0.030	11.970	< 0.001
WP	Mathematical Literacy	0.132	0.035	3.738	< 0.001
PT	Mathematical Literacy	0.099	0.044	2.259	0.024
PMR	Mathematical Literacy	0.104	0.036	2.914	0.004
AMR	Mathematical Literacy	-0.174	0.032	-5.386	< 0.001

This research studied the relationship among OTL, perceived control, and mathematical literacy. OTL itself is conceptualized as the frequency of engagement in four different mathematical tasks that was perceived by students. It is critical to think about the possible explanations for this finding, though verifying the speculated reasons is beyond the scope of this study. Here, we provide possible explanations that may be implemented by subsequent studies and, by extension, identify pathways for future research. First, we investigated the relationship between mathematical tasks as OTL and two types of perceived control: internal and external. On the one hand, students' OTL with procedural tasks was positively related to internal perceived control ($p < 0.001$); on the other hand, OTL with applied mathematics reasoning tasks was related to external perceived control ($p = 0.034$). Although these significant relationships were not necessarily causal, this suggests that students' experiences with different types of tasks in mathematics classrooms are one of the factors that shape students' perceptions of perceived locus of control.

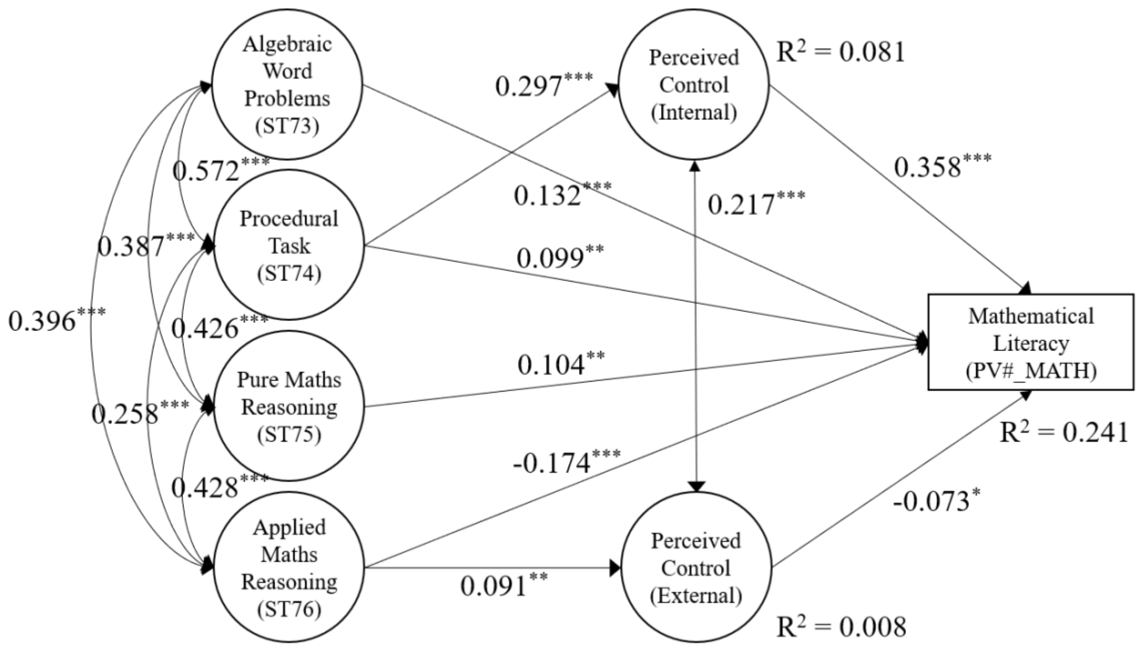


Figure 3. The SEM Results based on the Nested Model

When speculating several reasons for why perceived control and OTL diverged as observed, we put forward how students' engagement in mathematical tasks can relate to variations in their teachers' implementation of the tasks. Specifically, there could be larger variances in ways to implement of applied reasoning tasks (involving high cognitive demands) than in procedural tasks (involving low cognitive demands). In this sense, students' OTL in applied mathematical reasoning tasks can also vary according to how teachers' implement those tasks, which includes how the tasks are presented and how students' learning is scaffolded by teachers. Since cognitive demand for tasks varies by the ways of teachers' task presentation and scaffolding, students might perceive that their success/failure of the learning tasks is subject to teachers' implementation and that their success/failure is out of their control. In contrast to the applied reasoning tasks, procedural tasks do not give as much space for implementation-variation, and hence, students' positive/negative experiences with procedural tasks are perceived not to be contingent on how they are implemented by teachers. In this study, students who were frequently engaged in procedural tasks were more likely to think that their success/failure is under their own control, which resulted in a strongly positive relationship between OTL in procedural tasks and internal perceived control.

In this study, we also investigated the relationship between OTL via mathematical tasks and mathematical literacy scores. In constructing the model in Figure 1, we conjectured that all types of OTL would be positively related to mathematical literacy, though the degree of the relationship may vary. Particularly, we expected that applied mathematical reasoning tasks would have a stronger positive relationship to mathematical literacy than the other three task types. The PISA assessment for mathematical literacy involved items that assessed capabilities in mathematical reasoning, and use of mathematical concepts and procedures in various real contexts. At the beginning of the study, we reasoned that many constituent capacities of mathematical literacy are utilized when students engage in applied mathematical reasoning tasks; however, the findings challenged this conjecture. Indeed, OTL via applied mathematical reasoning was negatively related to mathematical literacy although all other mathematical tasks (word problems, procedural tasks, and pure mathematics reasoning tasks) were positively related to mathematical literacy. This means that those students who were exposed more frequently to applied mathematical reasoning tasks were likely to have lower mathematical literacy scores, whereas those who were more frequently engaged in other types of tasks were likely to have higher mathematical literacy scores.

At this point, we will discuss why OTL via applied reasoning mathematical tasks had a uniquely different relationship with mathematical literacy compared to the other three tasks. We will highlight the specific cognitive processes that are required to successfully engage in each type of task, and how we can characterize such thinking processes. According to Hanna and Jahnke (2007), engagement in mathematical tasks requires students to undergo two reasoning processes: (1) making abductive inferences, such as "an action of *selection,*" to build the correct premise and (2) making deductive inferences between the premise and the conclusion (p. 149). As presented in the example tasks (Tables 3 and 4), pure mathematical reasoning, word problems, and procedural tasks offer the correct premises directly to the students, thereby placing less emphasis on abductive inferences. For applied mathematical reasoning, abductive inference is one of the most important elements when formulating a given real situation and building premises.

Also, theoretically, mathematical literacy is defined as a combination of both abductive and deductive inferences, which includes the abilities of formulating premises, employing mathematical concepts and ideas, and interpreting solutions in real situations. However, the mathematical literacy that was measured in the PISA 2012 might not have captured both abductive and deductive inferences in a balanced way. Even though the OECD reported that PISA mathematics assessments are improved by using computer-based delivery formats, it is still difficult to evaluate students' inductive/abductive reasoning skills with mathematics test items. Students are asked to answer multiple-choices questions in the assessments, and this type of assessment does not well reflect students' processes of searching for reasons. Thus, in standardized test settings, often with time limits, students focus more on finding a correct answer from the information that is presented in the test problems, rather than exploring and formulating a real-word problem situation.

The questionnaire in PISA 2012 asked students how often they encountered each type of tasks during their mathematics classes. The negative relationship between applied reasoning mathematical tasks and mathematical literacy encouraged us to rethink how the frequency of OTL via applied reasoning tasks can affect mathematics learning. Departing from the idea of 'the more, the better,' we speculated that the implementation of applied reasoning tasks has much to do with how they are implemented, as opposed to how often they are implemented. Considering that the PISA student survey was about how often students encountered each type of tasks, it is possible that other important aspects of OTL, such as quality and process of OTL, were not taken into consideration. The varying quality of students' OTL of applied mathematical reasoning could be another reason for the negative relationship between applied mathematics reasoning tasks and mathematical literacy. Particularly, some researchers (e.g., Boston & Smith, 2009) have argued that teachers tend to reduce the original cognitive demands of mathematical tasks when implementing them. This means that the result is probably due to the ways in which those reasoning tasks were implemented. It can be challenging for teachers to scaffold students' learning process carefully and successfully by engaging them in applied mathematics reasoning tasks. As such, this may inhibit students from fully taking advantage of OTL via applied mathematical reasoning.

The findings of this study support that allocating more learning time to applied reasoning task is not necessarily beneficial to, or does not guarantee, overall mathematics learning. However, our attempts made so far to explain the *negative* relation between applied reasoning tasks and mathematical literacy still may not seem to be sufficient. Thus, future research on teachers' implementation of applied mathematics reasoning tasks in classrooms should be followed to validate and explain the negative relationship between applied mathematics reasoning task and mathematical literacy.

Another research question of our study was on the role of perceived control in the relationship between OTL with mathematical tasks and mathematical literacy. The results showed that engagement in OTL with procedural tasks is likely to influence mathematical literacy directly and indirectly through internal perceived control. Particularly, the effect of engagement in procedural tasks on mathematical literacy is even greater when the indirect effect through internal perceived control is taken into consideration. Considering the strong positive relationship between internal perceived control and mathematical literacy, students are likely to have high mathematical literacy scores when they believe that being successful in mathematics is under their control. To synthesize, OTL through procedural tasks is likely to promote students' internal perceived control, and in turn, this can have an effect on better mathematical literacy. Though this may suggest the merit of engaging students in procedural tasks in relation with students' perceived control, we are not to argue that procedural tasks should be offered more in mathematics classrooms than other types of tasks. As Yeo (2007) argued, students need to have a variety of OTL, from procedural tasks to mathematizing tasks, and teachers need to be cognizant about different OTL that is afforded by various types of tasks. This is specifically because OTL through different types of tasks may have varying effect on cognitive and non-cognitive processes during mathematics learning, as shown in our study.

CONCLUSION

This research showed that students can improve their mathematical literacy by engaging in various types of tasks from procedural tasks, word problems, to pure and applied mathematics reasoning tasks. Opposite from our expectation, the results showed that students are likely to have lower mathematical literacy when they have encountered applied mathematics reasoning tasks more frequently. In addition to the discussion of the results, we suggest future studies about how different types of tasks are implemented in classrooms, how such implementation influence students' perceived control, and how students perform on tests based on their classroom experiences.

We suggest several implications based on the findings in this research: First, teachers and curricular developers need to implement various tasks considering their relation to students' perceived control and mathematical literacy. However, it is important to recognize that students' frequent engagement in certain tasks could have unexpected influences on their mathematical literacy, especially when they are not appropriately facilitated. Particularly, Korean students' OTL with applied mathematics reasoning tasks had negative relationship with their achievement. These findings call for more investigation on how to implement such tasks appropriately.

Secondly, it is critical to consider how teachers select OTL with different types of tasks to offer in their teaching practices. When educators emphasize tasks with high cognitive demands, it is sometimes misunderstood that tasks with low cognitive demand are less beneficial to students' higher order thinking processes in mathematics learning. Even worse, tasks with low cognitive demand are considered as something that teachers should avoid. However, our findings show that procedure tasks can help students believe that their success is under their own control, which could lead to better mathematics learning behaviors and higher mathematical literacy through appropriate scaffoldings for other types and levels of mathematical tasks. Then, students will be able to engage in different types of mathematical thinking and perceive that they can succeed in mathematics learning with their own effort.

We did not make direct relation between educational contexts of Korea and the results of this study in our interpretation of the results, which should be noted when attempting to generalize the findings to different educational systems. Moreover, we highlight that mathematical literacy defined and measured by the PISA could be different from achievement measures in other mathematics assessments. Therefore, replication studies using other assessments tools or in other educational contexts are needed.

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INDONESIAN MATHEMATICS TEACHERS' KNOWLEDGE OF CONTENT AND STUDENTS OF AREA AND PERIMETER OF RECTANGLE

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Abstract

Measuring teachers' skills and competencies is necessary to ensure teacher quality and contribute to education quality. Research has shown teachers competencies and skills influence students' performances. Previous studies explored teachers' knowledge through testing. Teachers' knowledge of the topic of area-perimeter and teaching strategies has been assessed through testing. In general, items or tasks to assess mathematics teacher knowledge in the previous studies were dominated by subject matter knowledge problems. Thus, it seems that the assessment has not fully covered the full range of teacher knowledge and competencies. In this study, the researchers investigated mathematics teachers' Knowledge of Content and Students (KCS) through lesson plans developed by the teachers. To accommodate the gap in the previous studies, this study focuses on KCS on the topic of area-perimeter through their designed lesson plans. Twenty-nine mathematics teachers attended a professional development activity voluntarily participated in this study. Two teachers were selected to be the focus of this case study. Content analysis of the lesson plan and semi-structured interviews were conducted, and then data were analyzed. It revealed that the participating teachers were challenged when making predictions of students' possible responses. They seemed unaware of the ordinary students' strategies used to solve maximizing area from a given perimeter. With limited knowledge of students' possible strategies and mistakes, these teachers were poorly prepared to support student learning.

Keywords: Knowledge of Content and Students, Mathematics Teacher, Area and Perimeter, Teachers' Skills and Competencies

Abstrak

Mengukur keterampilan dan kompetensi guru diperlukan untuk memastikan kualitas guru dan berkontribusi pada kualitas pendidikan. Penelitian ini menunjukkan bahwa komptensi dan keterampilan guru mempengaruhi performa siswa. Penelitian sebelumnya telah mengkaji pengetahuan gru melalui tes. Pengetahuan guru pada topik keliling-luas dan strategi pembelajaran juga telah dikaji melalui tes. Pada umumnya, banyaknya soal pada tes didominasi oleh soal-soal tentang pengetahuan subjek yang diajarkan. Oleh karena itu, asesmen seperti ini belum mencakup kesuluruhan pengetahuan dan kompetensi guru. Pada studi ini, peneliti menginvestigasi pengetahuan guru matematika tentang KCS pada rencana pelaksanaan pembelajaran yang mereka kembangkan. Untuk mengakomodasi kesenjangan pada penelitian sebelumnya, penelitian kali ini berfokus pada pengetahuan tentang konten dan siswa (KCS) pada topik keliling-luas pada rencana pelaksanaan pembelajaran. Dua puluh Sembilan guru matematika yang sedang mengikuti pelatihan peningkatan kompetensi secara suka rela mengikuti penelitian ini. Dua guru matematika menjadi fokus penelitian studi kasus ini. Konten analisis dan interview semi terstruktur dilakukan dan datanya dianalisis. Terungkap bahwa peserta ini mengalami tantangan dalam memprediksi kemungkinan respon yang diberikan siswa. Mereka belum menyadari strategi siswa yang biasanya digunakan untuk menyelesaikan persoalan memaksimalkan luas dari keliling yang ditentukan. Dengan pengetahuan yang terbatas pada kemungkinan strategi siswa dan kesalahan siswa, guru ini kurang siap dalam mendukung siswanya

Kata kunci: Pengetahuan tentang Materi dan Siswa, Guru Matematika, Luas dan Keliling, Keterampilan dan Kompetensi Guru

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Shulman (1986) refers to Pedagogical Content Knowledge (PCK) as the ways of representing and formulating the subject that is understandable to others. Research have shown that student achievements are more affected by PCK than Subject Matter Knowledge (SMK) as the quality of instruction is related to PCK (Baumert et al., 2010; Hill, Rowan, & Ball, 2005; Hill, Ball, & Schilling, 2008). As the use of SMK terminology varies, SMK in this paper refers to common content knowledge (CCK) which is part of SMK (see Figure 1).

> *Hill, Ball and Shilling (2008), in seeking to conceptualize the domain of effective teachers' unique knowledge of students' mathematical ideas and thinking, proposed the following domain map for mathematical knowledge for teaching (see Figure 1) (White, et al., 2012, p.394).*

One specific aspect of PCK is the Knowledge of Content and Students (KCS). KCS is 'knowledge that combines knowing about students and knowing about mathematics (Ball, Thames, & Phelps, 2008, p. 401). It consists of anticipating what students are likely to think about, what they could find confusing or complicated, and what students are expected to do mathematically to complete the chosen task.

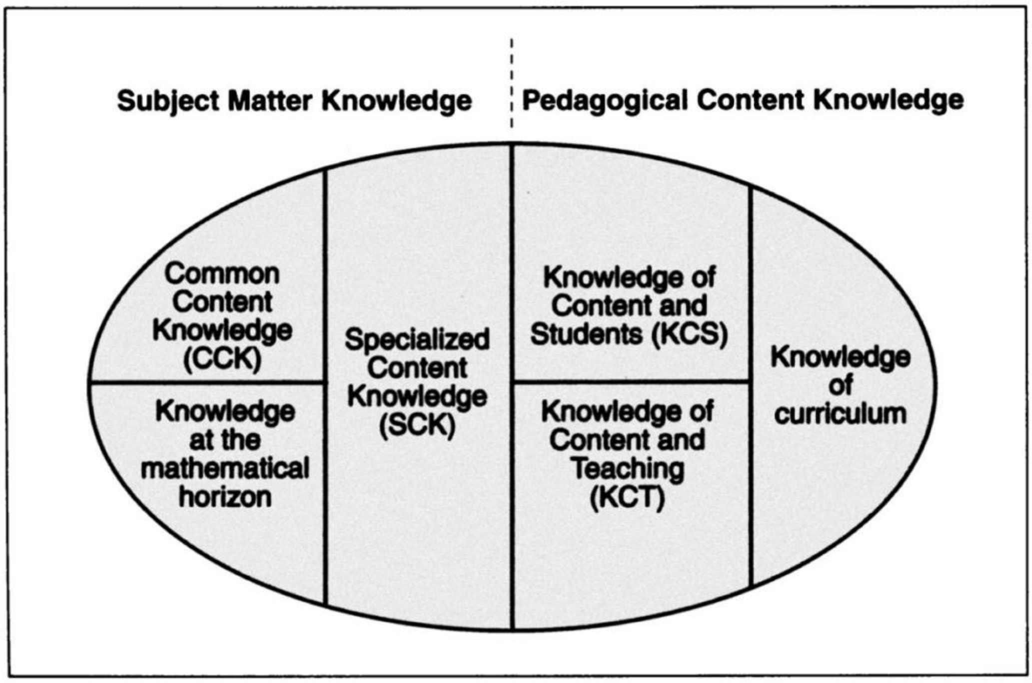


Figure 1. Domain map for mathematical knowledge for teaching (Hill, Ball, & Schilling, 2008, p. 377)

There are some teacher assessment models which measure knowledge for teaching. The Teacher Education and Development Study in Mathematics (TEDS-M) is one of the international assessments intended for pre-service mathematics teachers (Tatto et al., 2012). Some researchers assert that the Assessment of Teachers' PCK could be done through micro-teaching (Setyaningrum, Mahmudi, & Murdanu, 2018; Ünver, Özgür, & Güzel, 2020). In the case of pre-service teachers, they have challenges with student's thinking, mistakes and responding (Korkmaz & Şahin, 2019; Setyaningrum et al., 2018; Ünver et al., 2020). It makes sense as they have limited teaching experiences or even have not taught yet. For in-service teachers, Baumert and Kunter (2013) developed instruments to measure teacher's professional competence (COACTIV). The COACTIV adopted the three main core knowledge CK, PCK and PK from Shulman's work and extended it.

As one of the ways, testing is used to assess teachers. The Ministry of Education and Culture (MoEC) of the Republic of Indonesia has also implemented Teacher Competency Tests (TCT) to evaluate teachers' knowledge. The result of this assessment is both to evaluate teachers and to provide support for them (Widodo & Tamimudin, 2014). However, the content of this assessment is commonly dominated by SMK, in this case within the mathematical problems. It seems that the PCK has not been measured fully through this wide assessment. Another study using testing faced challenges in measuring teachers' knowledge (Fauskanger, 2015). An interesting finding of a study of pre-service teachers is that they possessed higher PCK scores than SMK from limited test items (Kristanto, Panuluh, & Atmajati, 2020). A case study in South Korea revealed that teachers with sufficient SMK of a certain competence/ topic faced challenges in incorporating KCS and KCT of that topic (Lee, Capraro, & Capraro, 2018). Therefore, testing to measure teachers' knowledge still face challenges.

Lesson plans are considered to play an important role in teaching and learning. Having a good lesson plan is important in ensuring that learning would take place during the lesson (Jones & Edwards, 2010). Academics argue that the key determinant of success in teaching is the effectiveness of planning and how well a plan is carried out in the classroom. Effective lesson planning considers possible classroom problems and how to tackle them adequately (Jones & Edwards, 2010). In the common Japanese lesson plan, it contains detailed instruction so that teachers can easily understand it when reading it (Nakahara & Koyama, 2000). Japanese lesson plans also include possible student solutions and errors. The blackboard is also carefully planned. Called, 'Bansho', which anticipates and tries to elicit student mathematical thinking and student thinking schema for solving the given problems.

In developing lesson plans, teachers integrate their knowledge, such as subject matter knowledge and pedagogical content knowledge (An, Kulm, & Wu, 2004; Burns & Lash, 1988; Simon, 1995). A study in Australia revealed that the teacher, in planning a lesson, gave attention to students' engagement (Clarke, Clarke, Roche, & Chan, 2015). The students' engagement involves a choice from many pedagogical strategies, all designed to motivate the students to engage with the topic. It has been shown by several studies that novice teachers improved their PCK by teaching and preparing to teach (Turnuklu & Yesildere, 2007). There is a reciprocal relationship between teacher thought process (including planning) and teachers actions, the latter much influenced by the former (Clark & Peterson, 1986; Superfine, 2008). In other words, teacher classroom practices are influenced by a complex mix of teacher beliefs, attitudes knowledge and intentions Therefore, arguably it is possible to look at teacher lesson plans to investigate their knowledge. The illustration of a model of teacher knowledge and planning can be seen in Figure 2.

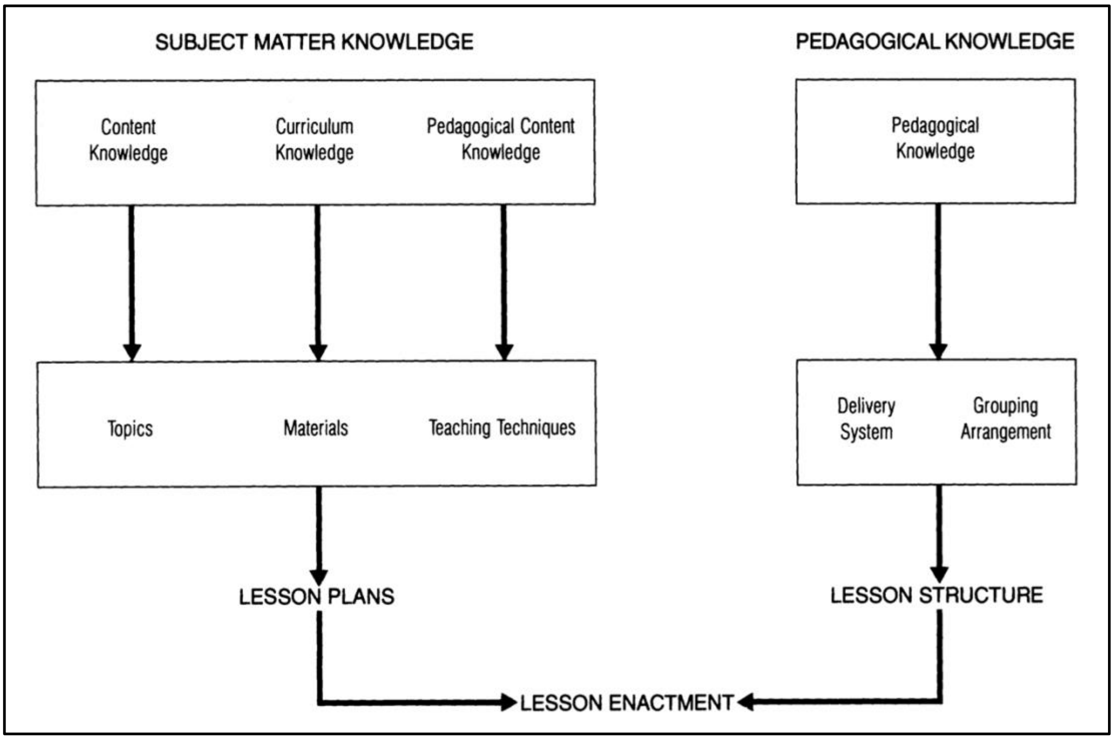


Figure 2. Model of teacher knowledge and planning (Burns & Lash, 1988, p. 382)

Carle (1993) has investigated several student misconceptions related to the area-perimeter topic. A meta-analysis of research has shown some student misconceptions on area measurement was due to area being taught together with perimeter causing many students to confuse area and perimeter (Watson, Jones, & Pratt, 2013; Cavanagh, 2007). Cavanagh (2007) studied Australian Year 7 secondary students and reported students experienced difficulties dealing with area concepts because of the above confusion with perimeter. As a consequence, students used slant and perpendicular height interchangeably. Zacahros & Chassapis, (2012) reported Greek Year 6 elementary students added the base plus the height instead of multiplying base with height to find the area of a rectangle. Özerem (2012) reported that seventh year secondary school students in Cyprus had a number of misconceptions due to a lack of knowledge related to geometry, resulting in them using the wrong formula. This lack of understanding of the concept of area resulted in students memorizing the formulas. Students who learn through manipulating area seem likely to avoid misconceptions on area measurement (Watson et al., 2013). It seems to make sense as they could manipulate and observe what changes happen by reshaping a figure (Yunianto, 2015).

It has been shown that SMK and PCK of mathematics teachers influenced student performance (Baumert et al., 2010). Thus, we should not expect teachers to deliver mathematics well if they do not have mastered it and do not understand how to teach it. Kow and Yeo (2008) explored the importance of SMK and PCK in the topic of area-perimeter from the planning of the lesson to its delivery. It was found that teachers with strong SMK and PCK provided more freedom to students to approach the task. Baturo and Nason (1996) evaluated first-year teacher education student understanding of subject matter knowledge in the domain of area measurement and uncovered many misconceptions. Success was related to their experience of learning the topic. John (2006) argued that novice teachers have difficulty making predictions about student responses and how to respond to unpredicted situations they encountered. In line with this, lack of mathematics pedagogical content knowledge of the teacher potentially lead to students having misconceptions (Kow & Yeo, 2008).

This study intends to focus on a part of PCK, the KCS within lesson plans on the topic of area-perimeter of a rectangle. It is necessary to obtain a fuller insight into teacher knowledge as it influence students' performance. Beside testing, there might be alternative way such as lesson plans to investigate teachers' knowledge. How are mathematics teachers prepare their lesson plans and how is PCK integrated in their lesson plans? How are the KCS integrated in the lesson plans? In the next section, the ways of gaining this insight will be discussed and the strategies used in collecting and analyzing the data. Furthermore, the results and discussion sections will describe the KCS evident in the lesson plans and the interviews with the respondents.

METHOD

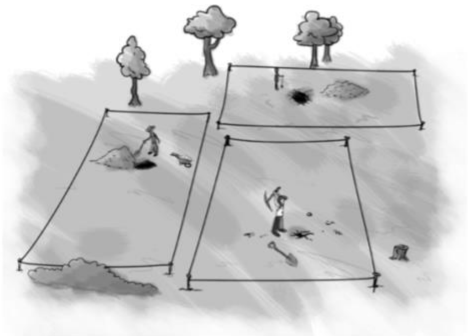
This research involved humans and has been approved by IOE research ethics of University College London (IOE.researchethics@ucl.ac.uk) as this is a part of completion of the first author's dissertation. This study administrated a case study approach. This approach suits this study as it does not seek to generalize the findings but to gain deeper insight into the issue (Denscombe, 2010; Yin, 2014). The research subjects were the mathematics teachers in Yogyakarta and its surrounding registered themselves to participate on PD organized by SEAMEO QITEP in Mathematics. Some teachers teach across multi-grades. The first researcher who was facilitating one of the sessions asked the participants to develop a lesson plan as part of the whole PD. It was done somewhere in the middle of all complete sessions. As it is a case study, the researchers examined two selected lesson plans of two mathematics teachers. The remaining lesson plans have not been analyzed due to time limitation. The sample was chosen from twenty-nine teachers who attended a professional development (PD) session, and two teachers were selected for the lesson plan analysis and interview. Additionally. these teachers were selected based on their teaching experience; at least five years. The interview scenario was a semi-structured interview, and the two teachers were interviewed together. The two teachers who had been interviewed were a female teacher and a male teacher. They have different years of teaching experience. The female teacher teaches in a city while the male teacher teachers in a rural area. Participation in this study was voluntarily. The Indonesian mathematics teachers attending this PD were teaching grade 7 to grade 9. The topic that would be taught was area and perimeter for grade 7. The "Gold Rush/Mining" task was selected. This task was chosen because it is a problem-solving task and has several ways to be solved on area-perimeter of a rectangle (see Figure 3). Additionally, the complete Gold Rush activity showed the mistakes that students might do. Thus, it is considered as a good activity to be explored to understand how teachers prepare this activity.

Gold Rush

In the 19th Century, many prospectors travelled to North America to search for gold.

A man named Dan Jackson owned some land where gold had been found.

Instead of digging for the gold himself, he rented plots of land to the prospectors.



Dan gave each prospector four wooden stakes and a rope measuring exactly 100 meters.

Each prospector had to use the stakes and the rope to mark off a rectangular plot of land.

1. Assuming each prospector would like to have the biggest plot, what should the dimensions of the plot be, once he places his stakes?

   Explain your answer.

Figure 3. The Gold Rush problem (https://www.map.mathshell.org/download.php?fileid=1637)

To analyze the lesson plans, the researchers used content analysis. This method has the 'potential to disclose many hidden aspects of what is being communicated through the written text' (Denscombe, 2010, p. 282). From the lesson plan, the researcher would investigate to what extent the teachers' knowledge of students' conceptions and misconceptions is reflected in their written lesson plans (Table 1). The two lesson plans were coded to find themes by classifying instructions and KCS integrated in the lesson plans.

Table 1. Knowledge of Content and Student (KCS) (Ball et al., 2008, p. 401)

No.	Knowledge of Content and Student
1.	The ability to anticipate what students are likely to think and what they will find confusing
2.	The ability to predict what students will find interesting and motivating when choosing a task
3.	The ability to anticipate how students are likely to solve a given task and whether they will find it easy or difficult
4.	The ability to hear and interpret students' emerging and incomplete thinking

By using Table 2, it is easy to differentiate instructions' categories. These themes were useful in providing information on what the lesson plans contained. It focused on whether or not, the teachers included information about what students would do to the task (KCS). The data were presented descriptively.

The two lesson plans were coded and analyzed. There were three types of instructions to refer to with the codes. First, general instruction (GI) is where the teacher gives students instructions in a general way. This type of instruction is relatively simple, short and contains the doer(s) and their actions (verb) but leads to some mysteriousness (unclear). The second type of instruction is specific instruction with no detail (SIND). This refers to specific action, which has more information than GI but lacks detail in necessary aspects. The last type of instruction is specific instruction with detail information (SID). This instruction provides more detail and clearer information. Some forms of SID are short and require no detail, as it can be found easily or understood easily in other parts of the text. Looking through the instruction types, the researcher seeks evidence of KCS on the lesson plans (Table 2).

Table 2. Coding for instructions

Code	Example 1	Example 2
GI	Teacher asks a question to students	Teacher asks students to present their work
SIND	Teacher asks a question to students about their strategy.	Teacher asks two groups to present their work
SID	Teacher asks a question to students about their strategy. "what did you do and How did you do it? How are you convinced with your strategies?	Teacher asks two groups with different strategies to present their work starting with the group with less sophisticated strategy.

The two teachers were also interviewed to gain more insight. They were interviewed together (focus-group interview). The researcher wanted to clarify what was written on the lesson plans and why. Through a semi-formal interview style, data were collected through voice recording as well as video recording. From the records, data were transcribed and analyzed.

RESULTS AND DISCUSSION

Using the codes, the lesson plans revealed some interesting findings. Teachers 1 (T1) and Teachers (T2) have different proportions of the use of the instructions (Table 3). The percentage is from type of instruction per total instructions written on the lesson plans.

Indonesian teachers follow the prescribed template of a lesson plan by MoEC. The template consists of three main parts namely; introduction, main and closure. It also consists learning goals and how teachers and students would do in the classroom.

Table 3. Proportions of the instructions

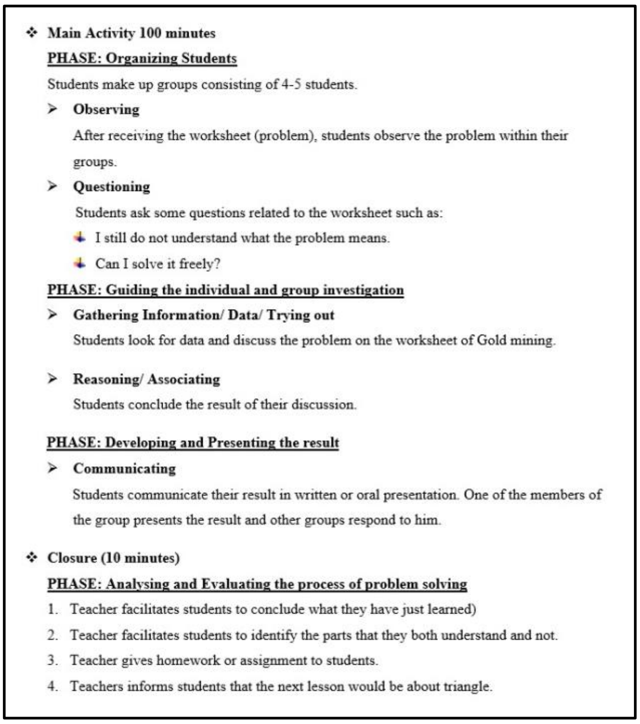
Instruction	**T1**	**T2**
GI	8 (35%)	6 (31.6%)
SIND	6 (26%)	7 (36.8%)
SID	9 (39%)	6 (31.6%)
Total	**23 (100%)**	**19 (100%)**

Based on the partition T1 used more instruction in the introduction and has less instruction in the main body. Interestingly, T2 has more instructions in the Main body with detailed information. Compared to T1, T2 had fewer total instructions, and detailed instructions (SID). From T2's SID, there were several instructions that provided information relating to PCK (Table 4).

Table 4. Comparison of Instructions

Code	**Introduction**		**Main**		**Closure**	
	T1	T2	T1	T2	T1	T2
GI	2	0	3	4	3	2
SIND	3	1	3	3	0	3
SID	7	2	1	4	1	0
Total	**12**	**3**	**7**	**11**	**4**	**5**

T1 put more details of what students would ask to her on her lesson plan. For instance: 'Can I solve it freely?' has been put on her lesson plan. This is a proof of PCK in the lesson plan, but not specific to KCS.



- **Main Activity 100 minutes**

PHASE: Organizing Students

Students make up groups consisting of 4-5 students.

- **Observing**

After receiving the worksheet (problem), students observe the problem within their groups.

- **Questioning**

Students ask some questions related to the worksheet such as:

- I still do not understand what the problem means.
- Can I solve it freely?

PHASE: Guiding the individual and group investigation

- **Gathering Information/ Data/ Trying out**

Students look for data and discuss the problem on the worksheet of Gold mining.

- **Reasoning/ Associating**

Students conclude the result of their discussion.

PHASE: Developing and Presenting the result

- **Communicating**

Students communicate their result in written or oral presentation. One of the members of the group presents the result and other groups respond to him.

- **Closure (10 minutes)**

PHASE: Analysing and Evaluating the process of problem solving

1. Teacher facilitates students to conclude what they have just learned)
2. Teacher facilitates students to identify the parts that they both understand and not.
3. Teacher gives homework or assignment to students.
4. Teachers informs students that the next lesson would be about triangle.

Figure 4. Teacher 1 Lesson Plan

In addition, the way she would organize the discussion are provided in detail. This would provide information to other readers/ teachers how the classroom discourse was managed (Figure 4). On the phase of guiding the individual and group investigation which be rich of KCS. In this lesson plan, detail ways of students might solve it or make mistakes and how to facilitate it have not been depicted.

The T2 lesson plan of rectangle using Gold Rush task depicted detailed information about a possible student strategy (KCS). Figure 5 shows that T2 considered one strategy that students would utilize by asking students to make a table. T2 prompted students to make a table and gave an example to start with simple numbers. Within that table students would investigate the largest area by filling the lengths and widths that added to 100. More interestingly, two examples with easy numbers were provided to support students. Therefore, T2's instruction can be understood as providing a method to solve the task, with much support given to students.

Main Activity

- ☑ Teacher divide students into groups
- ☑ Teacher delivers the worksheet to be discussed
- ☑ Teacher facilitates the learning processes

- For the first question, students are asked to make a table by filling up the length column and determine the width to make 100 m.
  for instance, p=10, l=… m
  p = 15 m, l = ...m, then the area = …. p=15 m, l= … m etc
  Students determine the largest area by themselves
- For the second question, after students have solved the largest area for one miner, then how if it is for 2 miners?
  Next, if the ropes of the 2 miners are joined, and continue like the first question, what will be the largest area?
  How if you continue doing this for 3 miners and 4 miners until n miners?

Figure 5. Teacher 2 Lesson Plan of Gold Rush

After finding the largest area of the rectangle, students had to find the largest area by joining two miners' ropes and how would they join it. T2 also offered questions for students, revealing the organization on their lesson plan. T2 has also provided students actions in Figure 6.

- Students evaluate and make generalisation into questioning.
  - ☑ Teacher asks students to present in front of the class
  - ☑ Other students respond the presenter

Figure 6. T2's lesson plan on organizing the classroom discussion

Students were expected to evaluate and generalize during discussion. Although it was unclear what kind of evaluations and generalizations would be made. It would be clear if he put, for instance, that the generalization would be that 'the largest area would always be a square'. This generalization might come out from students. In addition, it was not clear how T2 would organize the presentation, or which group would present first. If there were two groups with different strategies or different conclusions, it is not clear how it would be organized.

Teachers T1 and T2 have more than five-years teaching experience each. Based on the questionnaire and interview, their schools are different in terms of location and students' background. These teachers themselves employed different abilities in solving the Gold Mining problem (Figure 3). From the conversation below, it seems that they have three correct strategies or less to solve it: T1-Ms. Excel integration and T2 -table, quadratic function and graph. However, there is a significant difference between the two teachers. T1 allowed the students to solve the task freely (students' own ways).

The interview with Teacher 1 showed that she has the ability to solve the problem.

R : Are there other ways T1?
T1 : Yesterday, I just did that one.
T1 : …just let students find the ways to solve it …. Then, I will let them know that there are some ways to solve it. I give that opportunity to students

This teacher (T1) would allow her students to approach the task in their own ways. However, T2 had a different way of letting students approach the task, providing only one strategy.

*T2 : To me, I could do it directly because **I already knew it** but to students if I want to students to learn it, **I make a table for them**. If the table is not made, students will find it difficult to solve it for students in my school.*
R : So, you (T2), induce them by using the table?
T2 : Yes, by the table.
R : What do you think, how many ways to solve it?
*T2 : To me, I did one way I know it directly it would be a square. **I knew it already**. But for students, **with table**, students will measure the perimeter, area, so if the length is 5, how long is the width, if the length is 10, how long is the width, and.., they will list it, this is how I let them learn. If I do not do it they will have no clue to solve it.*

From the transcript of T2, he seemed to only allow his students to use one strategy. He believed that his students would not be able to approach the task without inducing the table. He has had previous experiences where students were unable to complete a similar task.

*T2 : I have tried several times an easier task, for instance, given the perimeter of a rectangle and how big is the area, changing from the perimeter to area, I let them do it and facilitated them, but students were not able. For the story problem, the reading comprehension, the task asks to go to the East, most of my students go to the West (**metaphor**).*
T2 : However, I have thought only one strategy, which is global to solve a task. … I, I... know at least I understand my students' characteristic so that it will be difficult for my students. … It is not possible to come up if I let them to do it freely. … I am so careful to give it the various strategies because students would get confuse

To know how to solve the mathematical task, these teachers tried the problem themselves. During the interview, T2 seemed to be familiar with the task and had three ways of finding the answer. Meanwhile, T1 only thought of one strategy.

T2 : By using the strategy of making rectangles with certain sizes and order them and estimate the biggest area.
T2 : To me, I did one way I know it directly it would be a square. I knew it already
T2 : ...instead of table, we can make the variable x, then I will be a quadratic function,
R : Are there other ways to solve it?
T2 : For the time being, not yet, making rectangles and to the square
R : Do you think there are still other ways to solve that problem?
T2 : I could use the graph ...

To some extent, from the lesson plan, T2 gave students a global strategy (table) to solve the task based on his previous experiences, although there is no guarantee that students would continue to have the same issues with the task (Figure 5). However, by giving the students the strategy, he inadvertently is making the students dependent on him. Whereas, from the lesson plan, T1 is helping the students to make decisions themselves (Figure 4). From the interview evidence, the two teachers have different abilities in solving the task and differ on the approaches they offer to their students.

In relation to students' possible mistakes and misconceptions, it seems that these teachers had some ideas as to what their students would find difficult.

T1 : The task has missing information, it should be more, and some students would think that. So that they ***have not thought*** *yet the possible ways to solve it. In average, students can directly solve it with possible ways to do. They can find it directly.*
T1 : 100. Maybe ***they thought that*** *that's the only think they know.*
R : ... So, they would answer it 100, possibly
T1 : Yeah, possibly
T2 : ... for those who did not understand, ***they would not know what 100*** *m rope is to with the perimeter. So that the concept of perimeter, for those who understood, they already make it but later* ***they would not think*** *the rectangles can be varied.*
T2 : Students ***would confuse*** *the meaning of maximum, which is the largest,* ***they have not thought about it.*** *So that students' thinking is not yet there. Their thinking is still circulated on the perimeter not yet the perimeter to area and from area to find maximum area.*

Teachers also have ways of responding to students' mistakes, prompted by the researcher (Figure 7). The researcher proposed a possible mistake by a student of which the shape looks like a rectangle 25 x 26,5.

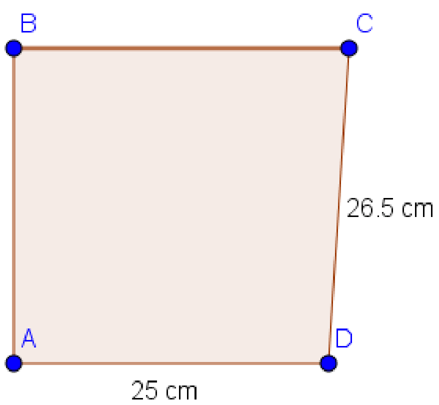


Figure 7. A student's possible mistake proposed by the researcher

If faced with a student mistake that they have not thought of before, both teachers seemed to engage thoughtfully with the scenario presented and sought ways of supporting students in addressing the mistake. Rather than telling a student their answer was incorrect, they asked what the task wants, and told them to check whether the shape is a rectangle or not.

R	*:*	*If it happens if you see this (showing)*
T1	*:*	*I would ask students back to try it then you calculate it as what being asked to you*
R	*:*	*They have not yet known the result!*
T1		*Try, try it, by trialing they would know that it is different, this one is more, and that one is like that, ...*
R	*:*	*T2, what if your students did this? what would you do?*
T2	*:*	*I would check it first, is it correct or not, the shape is a rectangle or not, they said that it is not, so I asked whether the perimeter is 100 cm or not. So, by knowing that it is a rectangle, the length would be equal, and the width would be equal (opposite sides), so that the perimeter would be 100 cm...*

In this study, the lesson plans facilitated an insight into teachers' knowledge. In this case, it showed teacher's pedagogical knowledge as well as PCK. Lesson plans can contain rich information on how the lesson is expected to be carried out. This is potential data to be used for assessing teachers' knowledge. How the teachers organize and manages the classroom, task, and the discussion would be depicted in the lesson plans. This resonates with Burns and Lash (1988) and Simon (1995) who argue that in developing lesson plans, teachers integrate their knowledge, such as SMK and PCK. On the other hand, experienced teachers may not use paper planning (written lesson plan) or just outlines as they have knowledge of what will work best (Butt, 2008; Jones & Edwards, 2010). In addition teachers also do mental planning for the lesson plans and the lesson plans are not written (Borko, Livingston, & Shavelson, 1990). The dynamics of a classroom are very fluid, and a teacher must adjust to that fluidity while following the plan. It is rare for a lesson to go exactly to plan. Yet, the execution of the lesson plan determines the effectiveness of the lesson (Kow & Yeo, 2008). In Japanese lesson plans, they contain more detailed instructions (Nakahara & Koyama, 2000) which shows more information about teachers knowledge. In contrast, the two case of teachers in this study, have not yet shown detailed instructions but more in general instruction.

Teachers have different ways of supporting students to solve tasks (Yeo, 2008). Students' performance is more affected from teachers' PCK (Baumert et al., 2010). However, SMK is basis knowledge for teachers (Shulman, 1986; Turnuklu & Yesildere, 2007). It is not usual that teachers teach 'something' before mastering the subject matter thus reducing the possibility of teaching effectively (Turnuklu & Yesildere, 2007). The teachers in this study were able to solve the task and had some ways to respond to students when they made mistakes in solving the given task (possessing SMK and PCK). However, these results are not generalizable. The limited sample was not chosen randomly and as these teachers came from relatively developed areas in Java and have at least five years teaching experiences they are not representatives of the wider Indonesian teaching population. Mathematics teachers in this study might not show detail information on their lesson plans and have not fully been aware of integrating PCK on developing their lesson plans. This study might not cover all mathematics teachers' PCK profile in Yogyakarta or broadly in Indonesia. However, this study has provided an interesting glimpse into one part of the very complex decision and knowledge processes that are involved in teacher pedagogical knowledge.

CONCLUSION

This study indicates that it is possible to assess teachers' KCS of a specific topic through analysis of the lesson plans when supported by interviews. There is evidence that these teachers had some knowledge about student strategies and misconceptions about the area-perimeter of rectangle topic, and that this knowledge was not necessarily fully integrated into their lesson plans. When prompted to think about possible misconception, the teachers found that it was challenging. Understanding possible misconceptions, making predictions and the anticipation of student responses would help teachers to be better prepared in facing the situations during teaching. Developing problem solving skills and autonomy among students requires teachers to stop providing a particular way (limiting students' strategies) but rather provide an environment where students are able to choose strategies, to make mistakes and to explore. Training for teachers could be more supportive in providing pedagogy that promotes such an environment. Additionally, this study explored a rectangle topic, the result might vary in different topics. Therefore, further investigation on different topic could be conducted. This study is not generalizable as it used limited research subjects.

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SECONDARY SCHOOL MATHEMATICS TEACHERS' PERCEPTIONS ABOUT INDUCTIVE REASONING AND THEIR INTERPRETATION IN TEACHING

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Abstract

Inductive reasoning is an essential tool for teaching mathematics to generate knowledge, solve problems, and make generalizations. However, little research has been done on inductive reasoning as it applies to teaching mathematical concepts in secondary school. Therefore, the study explores secondary school teachers' perceptions of inductive reasoning and interprets this mathematical reasoning type in teaching the quadratic equation. The data were collected from a questionnaire administered to 22 teachers and an interview conducted to expand their answers. Through the thematic analysis method, it was found that more than half the teachers perceived inductive reasoning as a process for moving from the particular to the general and as a way to acquire mathematical knowledge through questioning. Because teachers have little clarity about inductive phases and processes, they expressed confusion about teaching the quadratic equation inductively. Results indicate that secondary school teachers need professional learning experiences geared towards using inductive reasoning processes and tasks to form concepts and generalizations in mathematics.

Keywords: Perception, Inductive Reasoning, In-Service Mathematics Teachers, Secondary School

Abstrak

Penalaran induksi merupakan hal yang penting di pembelajaran matematika untuk membangun pengetahuan, pemecahan masalah, dan membuat generalisasi. Namun, baru sedikit penelitian yang telah dilakukan tentang penalaran induksi yang diterapkan di pembelajaran konsep matematika di sekolah menengah. Oleh karena itu, studi ini mengeksplorasi persepsi guru di jenjang sekolah menengah tentang penalaran induksi dan menjelaskan tipe penalaran matematika di pembelajaran persamaan kuadrat. Data dikumpulkan dari kuesioner terhadap 22 guru dan dari interview untuk mendapatkan jawaban lebih dalam. Melalui metode tematik analisis, ditemukan bahwa lebih dari separoh guru ini memahami bahwa penalaran induksi adalah suatu proses dari hal khusus ke umum dan sebagai cara untuk mendapatkan pengetahuan matematika melalui bertanya. Dikarenakan guru ini memiliki kejelasan tentang fase penalaran induksi dan prosesnya, mereka mengalami kebingungan tentang mengajarkan persamaa kuadrat secara induksi. Hasil penelitian ini menunjukkan bahwa guru di jenjang sekolah menengah ini membutuhkan pengalaman pembelajaran professional tentang penggunaan proses penalaran induksi dan penugasan untuk membangun konsep dan generalisasi di matematika.

Kata kunci: Persepsi, Penalaran Induktif, Guru Matematika, Sekolah Menengah

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The processes of the knowledge discovery and construction of proofs in mathematics involve both inductive and deductive reasoning (Davydov, 1990; Lee, 2016). The first implies moving from the particular to the general, and the second moves from the general to the particular (Hodnik & Manfreda, 2015). This work focuses on inductive reasoning, although some teachers are accustomed to employing deductive reasoning to teach mathematics (Rott, 2021). Siswono, Hartono, and Kohar (2020) defined deductive reasoning as 'a process of deducing conclusions from known information (premise) based on formal logic rules, where the conclusions must come from information provided and do not need to validate them with experiments' (p. 419). Inductive reasoning, on the other hand, is oriented to infer laws or general conclusions through observation and connection of particular instances (be they facts, premises, or particular cases of situations or a class of mathematical objects), and the conclusions can be verified by experimentation (Haverty, Koedinger, Klahr, & Alibali, 2000; Polya, 1957). According to Reid and Knipping (2010), three invariant characteristics of this type of reasoning are that it (a) comes from specific cases to conclude general rules, (b) uses what is known to conclude something unknown, and (c) is only probable but not certain.

Inductive reasoning has a core function in intellectual processes development for mathematics (Klauer & Phye, 2008; Mousa, 2017; Tomic, 1995). This type of reasoning is particularly important for learning mathematics in primary and secondary school, for two reasons. Firstly, it constitutes a teaching pathway for developing concepts and solving mathematics problems (e.g., Molnár, Greiff, & Csapó, 2013; Christou & Papageorgiou, 2007; Sriraman & Adrian, 2004). Inductive reasoning contributes to the formation of concepts because it 'lead[s] to detecting regularities, be it classes of objects represented by generic concepts, be it common structures among different objects, or be it schemata enabling the learners to identify the same basic idea within various contexts' (Klauer, 1996, p. 53). Secondly, it is one of the forms of reasoning that supports the process of generalizing numerical and figural patterns or mathematical objects (Cañadas, Castro, & Castro, 2008; 2009; Rivera & Becker, 2016).

The National Council of Teachers of Mathematics [NCTM] (2014) established that this form of mathematical reasoning must progress in students throughout each education level so that they can become more proficient in formulating conjectures and generalizations from specific cases. For that reason, secondary school teachers should develop and interpret the students' reasoning (AMTE, 2017; NCTM, 2020). However, several studies have reported that pre-service and in-service teachers have difficulties in solving generalization tasks from particular cases through inductive reasoning (Rivera & Becker, 2003; 2007; Sosa & Aparicio, 2020). In particular, they show difficulties associated with establishing a pattern and achieving the abstraction of the general when solving quadratic pattern tasks (Manfreda, Slapar, & Hodnik, 2012; Sosa, Aparicio, & Cabañas, 2019; 2020).

In this sense, knowing the type of perceptions that teachers have about inductive reasoning and how they promote it in teaching is essential to address these difficulties. Some studies suggest that promoting and interpreting inductive reasoning in the classroom is a complex task for teachers. Herbert, Vale, Bragg, Loong, and Widjaja (2015) reported that elementary school teachers have little understanding of the distinctive aspects of the mathematical reasoning types and how to encourage mathematical reasoning in the classroom. Furthermore, noticing and interpreting the actions of students' reasoning in generalization tasks is complicated for both pre-service and in-service teachers (Callejo & Zapatera, 2017; El Mouhayar, 2018; Melhuish, Thanheiser, & Guyot, 2018). De Koning, Hamers, Sijtsma, and Vermeer (2002) claimed that elementary school teachers have difficulty focusing on the inductive process when teaching mathematical structures because attention is paid to the content or to students' responses and not to the process itself.

Rott and Leuders (2016) reported that teachers have the epistemological belief that inductive reasoning justifies discovery in mathematics over deductive reasoning at a 2:1 ratio. Recently, Rott (2021) showed that an inductive belief prevails over a deductive belief in more than half a group of secondary school teachers, but those teachers did not provide arguments for their belief. According to this author, it is necessary to investigate the consequences of such epistemological beliefs in the mathematics teaching. To provide information in this direction, the aim in our study was to analyse secondary school teachers' perceptions about inductive reasoning associated with their interpretations of this reasoning in teaching the quadratic equation.

Negative, or inadequate, perceptions of teachers concerning mathematics could unfavourably affect students' learning (Rosli et al., 2020). Therefore, our study contributes to identify whether the teachers' perceptions about inductive reasoning are adequate or not to encourage this type of reasoning in their students. It is desirable that teachers have clarity about inductive reasoning phases that go along with the transition from the particular to the general for discovering properties, knowledge, and general rules in mathematics.

In this regard, some authors have pointed out the phases and inductive processes people use to generalize from particular cases. Polya (1967) proposed four phases: observation of particular cases, conjecture formulation, generalization, and conjecture verification. Cañadas and Castro (2007) developed an empirical model of secondary school students' inductive reasoning that expands the phases referred to by Polya and comprises the following seven phases: working with particular cases, organisation of particular cases, search for and prediction of patterns, conjecture formulation, generalization, and demonstration. Sosa, Aparicio and Cabañas (2019) reported that mathematics teachers managed to generalize inductively when they connected three cognitive processes: observation of regularities, the establishment of a pattern, and generalization formulation.

We assume that if teachers have inadequate perceptions or little understanding of inductive reasoning, they will have difficulty in promoting this reasoning in teaching. Besides, there is a gap in the literature concerning the secondary school teachers' perceptions about inductive reasoning, even when this type of reasoning is a means of mathematical learning and it is possible to develop it starting in elementary school (Molnár, 2011; Molnár, Greiff, & Csapó, 2013; Papageorgiou, 2009). These factors led us to ask: What are secondary school teachers' perceptions of inductive reasoning? And how do they interpret it in teaching the quadratic equation concept?

METHOD

This research is qualitative, exploratory, and interpretative. It is exploratory because perception and teaching of inductive reasoning of in-service teachers is a little-studied topic. There are only a few approaches to this topic from a cognitive perspective or from teachers' epistemological beliefs in the literature. An interpretative approach was considered to generate categories of teachers' perceptions and to identify ways in which teaching is carried out by interpreting and making sense of the characteristics attributed to inductive reasoning by teachers in a written and an oral way (Freitas, Lerman, & Park, 2017). The collection of data on the perception and interpretation of the teachers was carried out with an open questionnaire and an interview, both written and oral.

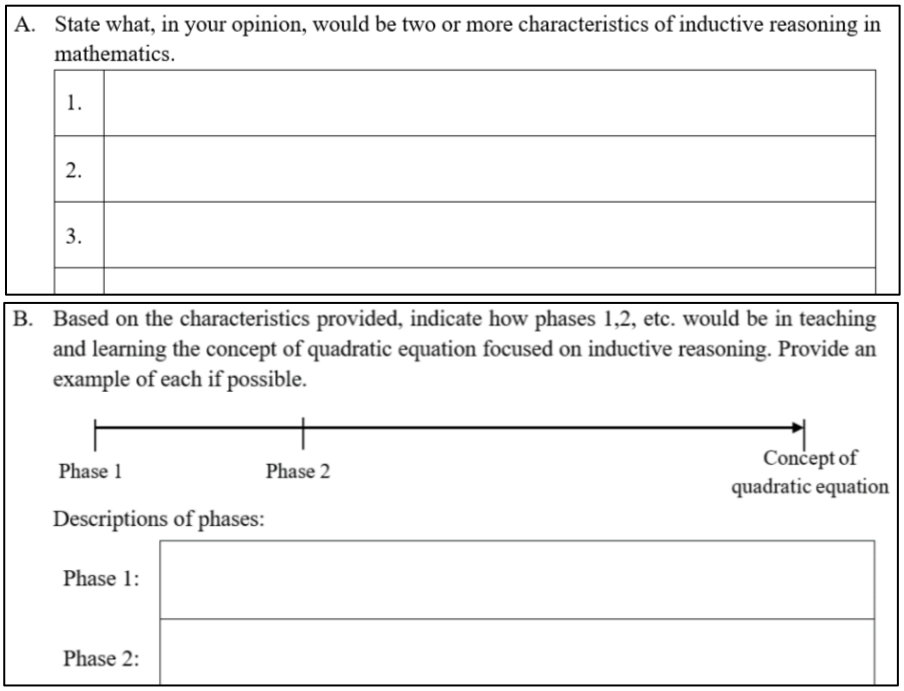
Context and Participants

This study was conducted with the participation of secondary school in-service mathematics teachers from Mexico; they were invited to participate in a professional teacher development program in mathematics through an open call. The program aimed to develop the teachers' inductive reasoning and to encourage them to enact this kind of reasoning in learning activities. Before the program began, 22 teachers—14 women and eight men—were selected among the teachers enrolled in the program; they agreed to participate in the study. The criteria for their selection were: (i) to have at least one year of experience teaching patterns and quadratic equations; (ii) to have the mathematical knowledge to solve tasks of generalization of quadratic patterns by inductive reasoning, whether acquired during their professional training or in training courses for teachers; and (iii) to know about inductive reasoning and mathematical generalization. The data for the selection of the participants were obtained from the academic information given by the teachers on the registration sheet for the program.

These criteria are explained by the fact that mathematics teachers have difficulties in generalizing quadratic patterns, as is reported in the literature. Besides, inductive reasoning is one of the mathematical reasoning types necessary to solve quadratic pattern generalizing tasks (Cañadas, Castro, & Castro, 2009; Rivera & Becker, 2016). The quadratic equation concept was chosen because, in the mathematics curriculum in Mexico, it is associated with the activity of generalizing quadratic patterns (Ministry of Public Education, 2017). In relation to the mathematical standards of the NCTM (2014), the aim of the mathematical activity in secondary school in Mexico (grades 7–9, ages 12–14) is to develop abilities such as generalization; abstraction; and inductive, deductive, and analogical reasoning. The students are expected to learn how to model linear, quadratic situations and to define patterns through algebraic expressions (Ministry of Public Education, 2017).

Data Collection

The data were collected in two working sessions. In the first one, the teachers gathered in a classroom and were asked to answer a written, open questionnaire, individually and simultaneously. This questionnaire was used to collect the responses of the participants about their perceptions of inductive reasoning and how these perceptions were brought to teaching. The questionnaire had two items, A and B, as shown in Figure 1. To obtain information about inductive reasoning perceptions, item A asked the teachers to write at least two characteristics of the reasoning in mathematics. Item B was oriented toward increasing understanding of how teachers interpret inductive reasoning to teach a mathematical concept. Therefore, item B asked participants to describe the phases to be followed to teach some aspect of the quadratic equation concept in an inductive way. The responses given by the teachers to this item were expected to be based on an interpretation of the characteristics mentioned in item A.


A. State what, in your opinion, would be two or more characteristics of inductive reasoning in mathematics.

1.	
2.	
3.	

B. Based on the characteristics provided, indicate how phases 1,2, etc. would be in teaching and learning the concept of quadratic equation focused on inductive reasoning. Provide an example of each if possible.

Descriptions of phases:

Phase 1:	
Phase 2:	

Figure 1. Questionnaire for data collection

Unlike closed or multiple-choice questionnaires, which contained predetermined responses given by the researcher that can skew the thinking of the study subjects, an open questionnaire contributes to a broader and more genuine picture of the perception of the participants (Ashton & Roberts, 2006; Peterson, 2000; Zohrabi, 2013). Thus, the teachers were asked to answer the written questionnaire to give them a greater opportunity to express themselves freely and to correct or complete their answers. There was no time limit for answering the questionnaire. This questionnaire was an adaptation of a previous questionnaire administered to a group of teachers with characteristics similar to those of the participants in this study to explore whether they knew the content (inductive reasoning and quadratic equation) of items A and B, and whether they understood what was requested.

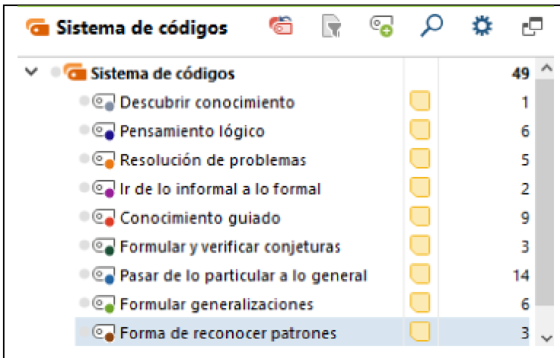
The participants were called for an interview in the second session. During the interview, one of the researchers (first author of this paper) posed questions in an individual and ordered manner to the participants about some words, phrases, or sentences that the teachers used in their responses to the questionnaire. The purpose of this interview was to expand, clarify, or verify their written information and to avoid ambiguities or inadequate interpretations of the written responses on the part of the researchers. The audio of the answers during these interviews was recorded and transcribed for the researchers' analysis, together with the data obtained from the questionnaire.

Data Analysis

A thematic analysis was conducted to describe the teachers' perceptions based on the written and oral answers to item A. The result of this analysis was the generation of categories of teachers' perceptions about inductive reasoning. Then, the responses given to item B were associated with these categories and contrasted with the conceptual framework to identify how teachers interpret inductive reasoning in teaching the quadratic equation concept.

The thematic analysis method consists of identifying, analysing, organising, and systematically obtaining patterns (themes) in a data set by detecting and making sense of the experiences and meanings shared in a group (Braun & Clarke, 2006; 2012). This method was used to identify patterns of meanings in the common characteristics that teachers attribute to inductive reasoning and to form categories related to their perceptions. The six phases of the method were as follows: 1) familiarising yourself with the data, 2) generating initial codes, 3) searching for themes, 4) reviewing potential themes, 5) defining and naming themes, and 6) producing the report (Braun & Clarke, 2012, pp. 60–69).

Phase 1 of the analysis consisted of repeatedly reading the written answers to item A and repeatedly listening to the audio with oral responses. This phase helped in developing an initial overview and making notes about the teachers' ideas concerning inductive reasoning. In phase 2, codes were assigned to extracts of written responses and audio transcripts with key phrases or with characteristics of the inductive reasoning mentioned by the teachers; nine codes were obtained (Figure 2). It should be clarified that a teacher's response could refer to different perceptions of reasoning. The response included more than one code in these cases, and therefore, the number of codified excerpts was larger than the number of participants. In phases 2, 3, and 4, MAXQDA (2018.2) software was used to encode data, group the excerpts of responses by codes to look for themes, and search for the ones with potential.



Code system:

- Discover knowledge
- Logical thinking
- Solving problems
- Go from informal to formal
- Guided knowledge
- Conjecture formulation
- Go from particular to general
- Formulate generalizations
- Way to recognise patterns

Figure 2. List of codes generated in MAXQDA (2018.2)

During phase 3, the generated codes and the excerpts associated with them were grouped and reviewed to search for themes that represent possible categories of the perception of the inductive reasoning of the teachers. For example, the codes 'way to recognise patterns,' 'formulate generalizations,' and 'formulate and verify conjectures' were grouped to form a category that refers to generalizations' formulation and verification.

In phase 4, the themes were recursively reviewed in the context of the codes and total set of responses. The members of the research team became involved in the review and exchange of information during the codification process, searching for themes and defining categories. The main author of this work carried out the first part of these processes in each phase. Another researcher reviewed the generated information later, and finally, the team came together to define the codes, themes, and final categories. In this way, during phase 5, the five categories concerning the teachers' perceptions about inductive reasoning were defined and named. Categories were defined by selecting excerpts of the responses to analyse, clarify, and exemplify each category and to name the resulting categories. Finally, a report for this paper (phase 6) was generated.

RESULTS AND DISCUSSION

The thematic analysis resulted in the detection of five categories of secondary school teachers' perceptions about inductive reasoning. These categories represent patterns of shared meanings among the participants, according to the characteristics that they attributed to this type of reasoning. The following sections present the title, a brief description, and some excerpts of representative responses for each participant perception.

Categories of Perception about Inductive Reasoning

Category A: Way to Acquire Mathematical Knowledge

Teachers perceive inductive reasoning as a pedagogical method of leading students to achieve new knowledge. For them, inductive reasoning consists of posing a problem and, based on the students' previous knowledge, asking key questions so that students acquire new knowledge, similar to the Guided Discovery learning (Honomichl & Chen, 2012). The following excerpts are examples of this perception:

Teacher C: *It involves the use of previous knowledge so that it can be applied in a more complex situation or to generate new knowledge.*

Teacher L: *Give students an exercise and, based on their previous knowledge, allow them to draw their own knowledge. Have students brainstorm to learn what they know.*

Teacher M: *One of the characteristics is to begin asking key questions for the exercises and introducing students to the topic. Students begin to reason about the topic through questions and can visualise the previous knowledge. Guide questions. During the class, doubts may emerge (...) and questions may be asked to reinforce the student's reasoning (...), students can achieve the appropriation of concepts, processes, etc.*

Teacher V: *Students can come to a conclusion or definition based on their ideas or previous knowledge.*

These were the teachers' predominant perception, though they differ from the function of inductive reasoning as a teaching pathway for the formation of mathematical concepts. The difference is that the teachers did not perceive the function of reasoning as recognising the particular characteristics or attributes of the concept from a set of situations and encapsulating it in a general attribute (Davydov, 1990; Klauer, 1996; Sosa, Cabañas, & Aparicio, 2019); instead, they described issues of the guided discovery so the students could organise and generate their own knowledge through interrogation and group discussion (Yurniwati & Hanum, 2017). This category shows that teachers do not perceive the relationship between the underlying cognitive processes of inductive reasoning and mathematical procedures as something central to the acquisition of new knowledge.

Category B: Cognitive Process

In this category, the teachers perceived reasoning as a process that allows moving from particular instances (e.g. ideas, particular cases, or specific situations) to infer a general conclusion or result. More than half the teachers revealed an adequate perception of inductive reasoning as a cognitive process that involves inferring laws or general rules through observation of particular instances (Haverty et al., 2000). The following excerpts show this perception:

Teacher B: *Start from particular cases to get to general cases. Other cases that meet the observed characteristics are obtained. Conjectures about the observed cases are formulated.*

Teacher E: *It goes from the particular to the general.*

Teacher N: *It is a type of reasoning that consists of moving from particular to general ideas. Starting from concrete ideas to ideas in general. Generalize based on experiences of the given results.*

This perception, very common among teachers, concerns an inherent characteristic of inductive reasoning: It goes from the particular to the general (Reid & Knipping, 2010). Teachers perceive the starting and ending point of inductive reasoning; generalization is recognised as an intrinsic element for this type of reasoning. However, they little or nothing allude to the specificity of the processes that allow continuous progress from the particular to the general; only a few participants described inductive phases or processes such as observing regularities, establishing patterns, and the formulation of generalizations (Polya, 1967; Sosa, Aparicio, & Cabañas, 2019).

Category C: Generalizations Formulation and Verification

Almost a quarter of the teachers associated inductive reasoning with the formulation of generalizations and referred to the experimental character of this reasoning to verify the produced generalizations (Polya, 1967; Soler-Álvarez & Manrique, 2014). That is, they perceive that inductive reasoning is associated with the ability to predict the overall behaviour of specific cases and verify their truthfulness.

Teacher responses referred to how to obtain a generalization and verify it, as can be seen in the following excerpts:

Teacher A: *[In inductive reasoning] students analyse certain characteristics that are repeated continually under specific conditions. It is that they achieve generalizations, establish some rule or generalization based on what is repetitive, verify the established statements (...) I believe that you can establish a statement, and it could be wrong, after the verification; if you said that it was continually happening, for example, for positive numbers, and something else happens for negative numbers(...), you must to prove that it is always repeating; but if you find a case that does not go in the same way, then the generalization will not work. It is like testing if this is real, if this is true.*

Teacher B: *After seeing specific and concrete cases, you can try to predict what is coming next—for example, in a sequence, make conjectures and try to prove them. Predict those conjectures, see if they can be proved, and finally, come to a generalization.*

Bills and Rowland (1999) argued that inductive reasoning is a means for producing mathematical generalizations from particular cases. Thus, Category C differs from Category B, in the sense that reasoning is characterised in terms of generalization as a product of the process of inductive reasoning (Klauer, 1990). According to Fernández-León, Gavilán-Izquierdo, and Toscano (2021), in-service teachers are used to informal reasoning (based on examples) in the justification and generalization (or conjecture) processes. This could be the case because the teachers have a slightly superficial perception of this reasoning type as a means for mathematical generalization. Therefore, inductive reasoning is only perceived globally as a means for predicting and proof in mathematics, but the punctual aspects of this reasoning are not considered. Broadening this perception could help teachers with the development and identification of mathematical conjectures based on empirical data (Cañadas, Deulofeu, Figueiras, Reid, & Yevdokimov, 2007).

Category D: Strategy for Solving Problems

In this category, teachers associate inductive reasoning with problem solving and perceive it as a strategy for obtaining and arguing for the solution. Some of the excerpts where this perception was identified follow.

Teacher P: *They are the premises that allow us to conclude the resolution of problems. It is the form of reasoning that allows us to argue the resolution of problems by induction.*

Teacher H: *Each student should try to solve the posed problem with his previous knowledge.*

Teacher O: *Considering hypotheses or some propositions as starting points to solve a problem.*

Teacher Q: *Establishing a process of resolution based on several cases or examples.*

Inductive reasoning is a useful strategy for solving mathematical problems of categorization, number series, similitude between objects and relationships, and generalization, among other problems (e.g., Csapó, 1997; Molnár, 2011; Tomic, 1995). Furthermore, it facilitates recognition of similitudes in the structure of mathematical problems and generalization of methods of resolution when students work with situations that have different contexts but the same underlying structure (Sriraman & Adrian, 2004). Nevertheless, as in the study of Herbert et al. (2015), the teachers seem to be less aware of the relationship between mathematical reasoning—inductive, in our case—and problem resolution than they were in the previous categories. In particular, we find that these teachers omitted the description of the inductive strategy for solving a mathematical problem; neither pointed out the potential of this reasoning for recognising methods of solving problems that have the same structure. As a consequence, this limited perception might not be enough to promote problem solving skills in students.

Category E: Logical Thinking

Some teachers perceived inductive reasoning as a part of logical thinking—that is to say, as a way of reasoning based on rules and the performance of orderly and coherent procedures. They mentioned the following relevant characteristics:

> Teacher J: *It emerges as part of a logical thinking process.*
> Teacher R: *Reasoning must be logical—I mean, in an orderly and coherent way. It must follow certain rules to carry out the exercises.*
> Teacher S: *It [inductive reasoning] is that the students develop logical thinking, that they understand what they do and perform the procedures in order.*

Category E suggests that the teachers must have a broader perception of inductive reasoning in logic such that they identify and establish inferences based on particular premises and recognise the probable character of the obtained conclusions or propositions (Hayes, Heit, & Swendsen, 2010; Reid & Knipping, 2010). This perception is associated with the fact that the teachers envision inductive reasoning as insufficient for validating mathematical propositions and believe that deductive proofs are needed (e.g. Conner, Singletary, Smith, Wagner, & Francisco, 2014; Martinez & Pedemonte, 2014).

The five categories of perception of inductive reasoning reveal that it is perceived in a very general way as a means and as an instrument to guide the acquisition of new knowledge and to make generalizations. However, the importance of the elements that constitute this form of mathematical reasoning is overlooked, specifically, the observation of regularities, the recognition of patterns, and the formulation of a generalization. Thus, while most of the teachers perceive generalization as a process and product of inductive reasoning, very few show clarities in this sense.

Interpretation of Inductive Reasoning in Teaching

The teachers' description of the phases for teaching quadratic equations led to the identification of four different interpretations of inductive reasoning in teaching. Two of these interpretations are associated with the perception of inductive reasoning as a way to acquire knowledge (Category A) and as a cognitive process to move from the particular to the general (Category B). The other two observed that the ways of teaching in the responses of the teachers are not inductive in nature; one of these forms belongs to deductive reasoning, and the other one was named iconic. Table 1 shows each interpretation and the number of teachers that expressed each interpretation.

Table 1. Inductive Reasoning Interpretations in the Teaching of Quadratic Equations

Interpretation	Frequency	Teachers
Way to acquire knowledge	8	C, E, F, K, L, M, S, T
Inductive (from the particular to the general)	4	B, I, N, R
Deductive (from the general to the particular)	6	A, D, O, P, U, V
Iconic	4	G, H, J, Q

Eight teachers' interpretation was that teaching a concept focused on inductive reasoning consists of guiding students to move from an existing or informal knowledge to a new knowledge, mainly through questioning or examples. This was the case for teacher M explained in Table 2.

Table 2. Phases for Teaching Quadratic Equation Proposed by Teacher M

Phase	Description
1	*Previous knowledge: Introductory questions about algebraic expression, algebraic language, power, the law of exponents.*
2	*Application of the concept of 'basic' shape areas (with square shapes).*
3	*Delete data and replace it with literals. Start with formulas.*

The phases proposed by teacher M are coherent with her perception of inductive reasoning as a way to acquire knowledge. She verbally emphasised the importance of starting with the previous knowledge of the students and using questions to guide them to the definition and expression of a quadratic equation:

> *Recover their previous knowledge; tell them that they had already worked with linear equations, but that there are other types of equations. After, I propose a daily life situation that leads students to represent a square; then I'm going to ask questions that guide them to the relationship of the figure with the formula of the area and make them pose the equation. Tell them that it is the quadratic equation.*

Certainly, inductive reasoning is a way to generalize knowledge by making inferences about unknown cases and new situations based on existing knowledge (Hayes, Heit, & Swendsen, 2010), but the way teachers consider incorporating it into teaching is not appropriate. Even when the teachers interpreted inductive reasoning as a way of teaching to acquire new knowledge, the phases proposed for teaching quadratic equations did not include inductive actions designed to recognise the structure of the equation in different situations or contexts that could allow for the identification of an essential quality of the concept (Davydov, 1990; Klauer, 1996), such as the quadratic behaviour of the variables.

A minority of participants (only four teachers) described the teaching phases in line with inductive reasoning. These phases involve actions concerning the observation of particular situations; the search for and recognition of invariant characteristics of the situations; and a generalization based on a formula, equation, or definition (Cañadas & Castro, 2007; Polya, 1967). For example, teacher B proposed four phases (Table 3) associated with the phases mentioned by Polya (1967). The phases indicated a way to move from the particular to the general, even when the teacher did not specifically refer to the quadratic equation or give examples to illustrate the phases.

Table 3. Teaching Phases based on Inductive Reasoning Described by Teacher B

Phase	Description
1	*Specific cases or situations that can be quantified, manipulated, or visualised are provided.*
2	*Different cases that meet the observed characteristic or property are proposed.*
3	*It is required to predict that this characteristic or property will be fulfilled for other cases that are not tangible or directly observable.*
4	*A rule or formula that covers all possible cases is obtained—that is, a generalization.*

Although the teachers perceived the transition from the particular to the general as a feature of this reasoning, the responses reveal a lack of clarity about the underlying processes. In this way, we consider that these teachers, like the elementary school teachers (De Koning et al., 2002), need instruction about questions or tasks that allow them to shift from the content per se and focus on the processes to develop inductive reasoning in the classroom.

On the other hand, six teachers evidenced confusion about teaching based on inductive reasoning, since the order of the proposed phases refers to deductive reasoning (from the general to the particular). That is, the first phase presents a definition, characteristic, or general formula of the quadratic equation, and the other phases lead to something particular, be it an example or the solution of a specific quadratic equation. Table 4 shows the responses of teacher V to illustrate this deductive interpretation in teaching quadratic equations.

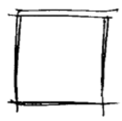
Table 4. Teaching Phases according to Deductive Reasoning Described by Two Teachers

Phase	Teacher V
1	*The characteristics of the quadratic equations are shown to the student.*
2	*Some examples are then shown; students will have to associate them with the quadratic equation of the form* $ax^2 + bx + c = 0$
3	*The student will have to find the unknown in the equation through factorisation.*
4	*a)* *First, the equation is written* $ax^2 + bx + c = 0$. *Example:* $3x^2 + 2x + 8 = 0$ *b)* *The expression is then factorised in linear factors.* $(3x - 4)(x + 2)$ *c)* *Set each factor to zero:* $3x - 4 = 0, x + 2 = 0$ d) *Find the value of* x: $x = \frac{4}{3}, x = -2$

Although teachers have a theoretical knowledge of inductive reasoning, it is insufficient to enable them to carry it out in their teaching practice; they are more familiar with teaching using a deductive approach than an inductive one. Even when pre-service and in-service teachers tend to believe that the discovery of mathematics knowledge is inductive (Rott, 2021), the common teaching sequences of some teachers are still in line with deductive reasoning.

The teaching phases proposed by four teachers did not differentiate between inductive and deductive reasoning; instead, they offered an iconic type of treatment. Table 5 shows the phases proposed by teacher G as an example of this type of teaching.

Table 5. Teaching Phases for Quadratic Equations Proposed by Teacher G

Phase	Description
1	*Starting with the area of a square, the student must use his previous knowledge. Area (A) equals side by side (l); area equals the square of the side.* $A = l \cdot l$ $A = l^2$
2	*For example, given the figure of a square, what is the length of the side of the square if its area equals 400 square metres? And if the area is 100 square metres?* 
3	*Make the figure bigger, adding different measurements to the sides of the squares so that they form rectangles or a bigger square.*
4	*Then write x instead of the measurement of the side of the square, so its area equals* x^2, *a squared number.*

In these cases, the phases involved the representation of a quadratic equation by a square or rectangular figure and, sometimes, a squared number. The teachers could have used this geometrical approach and inductive reasoning to obtain a general property: Every quadratic equation may be expressed as the product of linear factors of its roots; but the teachers focused on associating the degree of the equation with the area of squares and rectangles.

The types of interpretation of inductive reasoning in the teaching of the quadratic equation are consistent with the categories of perception. Although the secondary school teachers perceived positive qualities of inductive reasoning in teaching and in the mathematical thinking of the students, most of the teachers in the group showed confusion or an inadequate interpretation of this reasoning when describing the teaching phases. Consequently, most of the teachers' interpretations are inadequate to foment the acquisition of the quadratic equation concept through this reasoning as they relegate the associated phases.

This could be the case because the teachers ignore the principles that guide the development of mathematical reasoning in the students based on generalizations and justifications in classroom (Mata-Pereira & da Ponte, 2017). In addition, the data suggest that the teachers are not aware of the processes (search for attributes and relationships, comparison of similitudes and differences among attributes, resolution and control) involved in the connection of knowledge in an inductive way (De Koning & Hamers, 1999; De Koning et al., 2002; Klauer, 1996).

CONCLUSION

This research identified five perceptions about inductive reasoning among secondary school teachers, which reflects that little clarity and sensibility are present regarding this type of reasoning in the teaching of mathematics. Although these perceptions are positive, teachers need to enhance their interpretations of inductive reasoning if they are to develop such reasoning in the classroom. Results suggest that it is necessary to confront and broaden secondary school teachers' knowledge about inductive reasoning to develop their teaching competency. In particular, it would be important for teacher learning and professional development programs to help clarify the use of this reasoning in the mathematical concepts' formation, along with recognising and articulating inductive processes in contexts of mathematical generalization and problem solving. In efforts to enhance understanding of the use of inductive reasoning in teaching, the results of this study could be used to investigate the relationship between the resolution and the use of tasks involving inductive reasoning by secondary teachers and the type of perceptions those teachers have.

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RELATIONSHIP BETWEEN MATHEMATICAL LITERACY AND OPPORTUNITY TO LEARN WITH DIFFERENT TYPES OF MATHEMATICAL TASKS

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Abstract

We investigated how the opportunity to learn (OTL) with different types of mathematics tasks are related to mathematical literacy and the role of perceived control in the relationship between OTL and mathematical literacy. The structural equation modeling was applied to the data of 1,649 Korean students from the PISA 2012 database. OTL with the four different types of tasks – algebraic word problems, procedural tasks, pure mathematics reasoning, and applied mathematics reasoning – were measured via student survey on how often they have encountered each type of task in their mathematics lessons and tests. The results showed that OTL with the procedural tasks was likely to increase mathematical literacy directly and indirectly through internal perceived control. Engaging in the applied reasoning tasks is positively related to external perceived control, but negatively to mathematical literacy.

Keywords: Opportunity to Learn, Mathematical Tasks, Mathematical Literacy, Perceived Control, PISA 2012

Abstrak

Kami menyelidiki tentang bagaimana Kesempatan Belajar (KB) siswa dengan berbagai jenis tugas matematika yang terkait dengan literasi matematika dan peran *perceived control* dalam hubungan antara KB dan literasi matematika. Pemodelan persamaan struktural diterapkan pada data 1.649 siswa Korea dari database PISA 2012. KB dengan empat jenis tugas, yaitu soal cerita aljabar, tugas prosedural, penalaran matematika, dan penalaran matematika terapan, yang diukur melalui survei siswa tentang seberapa sering mereka menjumpai setiap jenis tugas dalam pelajaran dan tes matematika mereka. Hasil penelitian menunjukkan bahwa KB dengan tugas prosedural cenderung meningkatkan literasi matematika secara langsung dan tidak langsung melalui *perceived control* internal. Keterlibatannya dalam tugas penalaran yang diterapkan, bernilai positif terhadap *perceived control* eksternal, namun bernilai negatif terhadap literasi matematika.

Kata Kunci: Kesempatan Belajar, Soal Matematika, Literasi Matematika, *Perceived Control*, PISA 2012

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Since Carroll (1963) introduced the concept of opportunity to learn (OTL), OTL has been conceptualized as the inputs and processes that are needed to produce student achievement of intended outcomes (Elliott & Bartlett, 2016). Building on the early conceptualization of OTL as allocation of learning time (Carroll, 1963; Cogan & Schmidt, 2015), studies on mathematical practices and OTL are concerned with the processes through which individuals come to know mathematics content (Barnard-Brak et al., 2018). A contemporary definition of OTL is comprised of factors that have a significant influence on teachers' instructional practices and students' learning (Stevens & Grymes, 1993); these factors include content coverage and emphasis. Content coverage refers to which concepts and cognitive skills of curricula are covered during classroom learning, whereas content emphasis is related to activities and tasks that engage students (Stevens & Grymes, 1993).

In this study, OTL in mathematics classrooms is conceptualized as mathematical tasks that allow learners to have actual experiences with mathematics, which focuses on OTL as content emphasis in learning tasks (Schoenfeld, 1992). Learners' cognitive processes are shaped through the experiences with learning tasks. Further, by engaging in the mathematical tasks, students develop their understanding of what it means to *do mathematics* (Schoenfeld, 1994). By tackling various types of mathematical tasks, students can do mathematics and construct an epistemological understanding of what doing so means. Therefore, we conceptualize OTL as cognitive processes that learners engage in while doing mathematics through engaging in different types of tasks.

OTL, which is conceptualized around mathematical tasks, is related to students' learning outcomes, according to the framework suggested by Stein et al. (1996). However, a large body of literature based on this framework has focused on teachers' instructional practices, such as teachers' implementation of mathematical tasks that require high cognitive demands. There are few previous research studies that exist on the relationship between students' OTL and their learning outcomes, and these studies' conceptualization of OTL did not include mathematical tasks used in the mathematics instruction. For example, OTL in the study of Ottmar et al. (2014) consisted of two dimensions: instructional quality (teachers' efforts to promote reasoning and understanding of concepts via teacher-student interaction) and exposure to mathematics instruction (how long students were exposed to mathematics instruction). To bridge this research gap, we question whether there is a relationship between OTL (students' exposure to different types of mathematical tasks) and mathematics achievement.

Due to the complex nature of learning environments (Berliner, 2002; Jacobson et al., 2019) we hypothesize that the relationship between our conceptualization of OTL (frequency and type of mathematics tasks) and mathematics achievement is not only a direct, but also an indirect relationship through other factors. According to the framework of mathematical instructional tasks (Stein et al., 1996), the mathematical tasks that are set up by the teacher interact with and shape students' dispositions, including attitudes, beliefs, and motivation. This interaction between mathematical tasks and learning disposition in turn, influences students' cognitive processes and learning behaviors (Henningsen & Stein, 1997). Finally, the overall processes that involve mathematical tasks and students' perception are reflected in the students' learning behaviors and outcomes. For this reason, we consider students' perceived control as a meditating factor between OTL and achievement. Students who believe that academic outcomes are under their own control are predicted to be more actively engaged in mathematical tasks and earn better academic outcomes. This interaction between learning tasks and students' disposition (specifically, perceived control in this study) has not been found in previous literature.

This study is a secondary analysis that uses the database from the Program for International Student Assessment (PISA) 2012. Using extensive data from the PISA 2012 database, we investigated the relationship between OTL, mathematical literacy, and perceived control using structural equation modeling (SEM) approach. In this study, we focused on one educational context, South Korea, rather than examining different contexts of multiple countries. Before making international comparisons, an exploratory study to understand the phenomenon in a single context would be required for meaningful conclusions. Moreover, the rationale for selecting South Korea is that it is one of the high achieving countries, and that it has not been fully investigated in terms of OTL (Son, 2012). Furthermore, the OECD working paper (Schmidt, Zoido, & Cogan, 2014) showed that in each country, there is linear or quadratic relationship between exposure to different types of mathematical tasks and mathematical literacy. Since we focus on Korean student data and include perceived control in SEM analysis, our study can provide a broader picture of the relationship among different types of mathematics tasks, perceived control, and mathematical literacy.

The purpose of this study is to explore the relationship between OTL – a combination of exposure and types of mathematical tasks – and mathematical literacy measured in the PISA 2012. The nature of the relationship between OTL and mathematical literacy is not direct, and rather, the relationship is mediated by perceived control. OTL is hypothesized to be related to mathematical literacy via students' perceived control.

Opportunity to Learn

Within an educational context, OTL refers to the inputs and processes that are provided to students for intended learning outcomes (Elliott & Bartlett, 2016). One of the first conceptualizations of OTL focused on sufficient time and adequate instruction to learn (e.g., Carroll, 1963; Schmidt, 1992). With growing interest in the concept of OTL in relation to the demand for curricular validity, the concept of OTL has been expanded to accommodate a multi-dimensional construct that encompasses both the quality of instruction and its alignment with the assessment of learning outcomes (Abedi & Herman, 2010). Specifically, Stevens (1996) proposed a comprehensive conceptual framework of OTL that includes four elements: content coverage, content exposure, content emphasis, and quality of instructional delivery. As such, OTL, as a comprehensive and multi-dimensional concept, offers a basis for investigating students' learning in the mathematics classroom (e.g., Abedi & Herman, 2010).

When considering how these different dimensions of OTL are realized in the mathematics classroom, it is clear that mathematical tasks serve as a critical learning space that provide students with experiences of mathematical practice. In other words, mathematical tasks that comprise different dimensions of OTL (e.g., content coverage, content exposure, content emphasis, and quality of instructional delivery) interact with and, in turn, shape students' learning processes, both cognitive and non-cognitive. For example, in the studies of Watson (2003) and Törnroos (2005), class tasks, in addition to the curriculum and the textbook, were identified as one of the critical aspects of OTL.

In the PISA 2012, OTL is conceptualized as a constellation of three constructs that describe classroom learning environments: (1) measurement of content, (2) teaching practices, and (3) teaching quality (OECD, 2013). According to Schmidt et al. (2014), OTL in the PISA 2012 refers to the content students learn, as well as the cohesiveness that exists between what is taught and what they actually learn. Also, students' experiences with mathematical content are shaped by instructional practices, including student-centered instruction and lectures. Students' OTL is characterized by the factors underlying the quality of instructional practices, such as classroom organization, emotional support, and cognitive activation (OECD, 2013).

In the frameworks that are used in previous studies and PISA 2012, the concept of OTL includes specific content that is covered in mathematics classrooms, as well as mathematical tasks that deliver mathematics content. On one hand, the commonality of these frameworks is that mathematical tasks are an important factor of mathematics learning, and that teachers can affect students' cognitive and motivational processes of learning by designing these tasks. On the other hand, we also recognize differences among the frameworks that conceptualized OTL. One difference between the PISA 2012 framework and other literature on OTL is that in the PISA, OTL is operationalized as students' judgment on whether and how often they have encountered different mathematical tasks. This operationalization for measurement is partly limited in covering depth of teaching or quality of instructional delivery variables (E.g., Stevens, 1993), which is also recognized in the PISA 2012 framework (OECD, 2013, p.187). As such, we do recognize the multifaceted characteristic of OTL, but also acknowledge that large-scale assessment would not be enough to fully understand OTL that students experience in mathematics classrooms as reported in the PISA 2012 framework. In this study, we assume the operationalization of OTL in the PISA 2012, student-reported frequency of being exposed to different types of mathematical tasks.

Mathematical Tasks

Among the multiple aspects of OTL, we highlight students' exposure to different types of mathematical tasks in lessons and tests, as the tasks themselves reflect what content the students learn and what doing mathematics entails (Stein et al., 1996). In other words, mathematical tasks are essential tools for ensuring that students can understand mathematical concepts more fully, as well as to develop cognitive processes of mathematical reasoning via their experience with the tasks (Martin & Gourley-Delaney, 2014).

With regard to the cognitive processes of learning, the students' experience of mathematics depends on the level of cognitive demands, how the tasks are presented, and how the tasks are implemented. Adopting the conceptual framework regarding the relationship between variables that are related to tasks and students' learning outcomes (Stein et al., 1996), many studies have shown that cognitive demands of mathematical tasks can change as they are implemented (e.g., Boston & Smith, 2009; Henningsen & Stein, 1997). When students engage in mathematics, their reasoning differs according to what type of mathematical tasks are being offered (see *Potential of the Task* in Boston & Smith, 2009). Mathematical thinking processes that students employ are closely related to the mathematical tasks that are embedded in the learning context (Henningsen & Stein, 1997). Certainly, the elements and characteristics of mathematical tasks require students to engage in different cognitive processes. Hanna and Jahnke (2007) provided a good example by comparing activities that involve either pure mathematics or a real-life situation. Proving statements is a combination of two processes: "(1) finding the 'right premises' and (2) devising the chain of deductive steps leading from the premises to the statement" (p. 149). Mulnix (2012) labeled these as the process of *searching for reasons* (e.g., abduction/induction) and the process of *giving reasons* (e.g., deduction). According to Hanna and Jahnke (2007), the process of giving reasons is more emphasized in tasks that involve pure mathematical reasoning, which is why the process of searching for reasons has usually been downplayed. In contrast, mathematical tasks with real situations require setting up the premise first (searching for reasons), which is followed by the process of building logical connectedness (giving reasons).

Previous studies have scrutinized mathematical tasks set up by teachers and teachers' actual implementation of the tasks based on the framework developed by Stein et al.'s (1996) (e.g., Arbaugh & Brown, 2005; Boston & Smith, 2009). However, few studies have been conducted to investigate the link between mathematical tasks and students' learning outcomes, particularly measured by large-scale assessments. This is possibly because of the assumption that large-scale assessments are designed to evaluate students' content knowledge, not their mathematical practices (Lane, 2004). However, we argue that students formulate and utilize epistemic and cognitive resources to reason through OTL with mathematical tasks (Hammer, 2000), and students utilize some of those resources to solve problems in assessments (Bailin & Siegler, 2003; Hwang et al., 2020). The common cognitive resources used while engaging in mathematics tasks and assessment settings can help us to understand how students' OTL is connected to achievement scores in large-scale assessments.

The relationships between OTL and achievement can also be influenced differ by what mathematics tasks are involved in students' OTL. Individual differences in mathematics learning can be understood as interaction between features of tasks and students' inputs (i.e., cognitive resources, affectivity; Bornemann et al., 2010; Muis et al., 2015). As discussed, students' mathematical reasoning, as one of the critical components of doing mathematics, differs by what type of mathematical tasks are offered to them. The emotional components, such as task valuing and perceived control (Muis et al., 2015), can motivate them to either continue reasoning or terminate the reasoning process (McLeod, 1992). Particularly, students' perceived control – "the tendency of people to perceive that outcomes in a particular arena were either within or outside of their control" (McNabb, 2003, p. 418) – influences students' approaches to solving mathematical tasks. For example, students are likely to engage more actively when they believe that the outcomes from engagement in tasks are under their control (Hrbáčková et al., 2012).

Perceived Control

Control beliefs refer to an overall set of beliefs about how effective one's process of producing expected outcomes can be (Skinner et al., 1998). In academic settings, perceived control is understood as a critical psychological disposition that affect students' behavior, emotion, and achievement (d'Ailly, 2003; Schunk, 1984; Murayama et al., 2013). According to the previous frameworks of perceived control (e.g., Skinner et al., 1998; Rotter, 1966; Rotter & Mulry, 1965), perceived control over learning is constituted of two types of beliefs: strategy beliefs (what it takes to do well) and capacity beliefs (whether I believe I have the strategies; Skinner et al., 1998). According to Rotter (1966), people differ in their beliefs whether outcomes occur independently of how one behaves (external control) or are highly contingent on one's behavior (internal control). The construct of locus of control assume that internal and external causes are inversely related to each other and thus, can be assessed as a single, bipolar dimension (Skinner et al., 1990). Though perceived control has been shown to be an important indicator of students' motivation in learning (Patrick et al., 1993), previous studies on perceived control rarely examined it in the relation to the success in academic tasks (Skinner et al., 1990; Lipnevich et al., 2016).

In PISA 2012, the conceptualizations of perceived control and other self-perceptions are based on the *planned behavior theory* of Ajzen (2002). According to Ajzen (2002), perceived control belief is conceptualized as a person's belief about the ease or difficulty of performing a behavior and this belief forms a behavioral intention that directly increases the likelihood of a desired behavior. In this study, we viewed locus of control as having two types and identified student survey items that ask about their locus of control over mathematics learning and categorized them into internal and external perceived control. It is assumed that a learner's strong internal perceived control does not necessarily lead to weak external perceived control, and furthermore, they are qualitatively different with various sources of beliefs.

Mathematical Literacy in the PISA

The relationships between OTL and achievement can differ by how achievement is defined, and with what measure it is assessed. In this study, achievement scores represent students' mathematical literacy measured with the PISA 2012. According to the PISA 2015 framework, mathematical literacy "explains the processes content knowledge, and contexts reflected in the assessment's mathematics problems", and this shows how students perform in mathematics (OECD, 2017). The construct of mathematical literacy describes competency of individuals to reason mathematically and use math concepts, procedures, facts, and tools to describe, explain, and predict phenomena (OECD, 2017). This conceptualization of mathematical literacy supports the importance of students' engagement in pure mathematics tasks (reason mathematically) and their exploration in the abstract world of mathematics (use math concepts, procedures, facts, and tools; OECD, 2017).

When contemplating PISA's definition of mathematical literacy, it also emphasizes the capacities to formulate problem situations, employ mathematical problems, and interpret mathematics results in various contexts. In other words, rich experiences of real-world tasks in math classrooms are essential in developing these capacities. Accordingly, having experiences of doing mathematics in real world contexts (personal, societal, occupational, and scientific situations) contributes to the development of mathematical literacy.

Hypothesized Model

According to the literature review, we suggest the hypothesized model in Figure 1 representing the relationships among OTL by different types of tasks, perceived control, and mathematical literacy. This model includes a latent variable for each type of mathematical tasks (word problems, procedural tasks, pure mathematics reasoning, and applied mathematics reasoning tasks. In addition, there are two latent variables for internal and external perceived control in the model. By evaluating the appropriateness of the hypothesized model, we aimed to answer the following questions: (1) what are the relationships between opportunities to learn with different types of tasks and perceived control? (2) what are the relationships between opportunities to learn with different types of tasks and mathematical literacy measured in the PISA 2012?

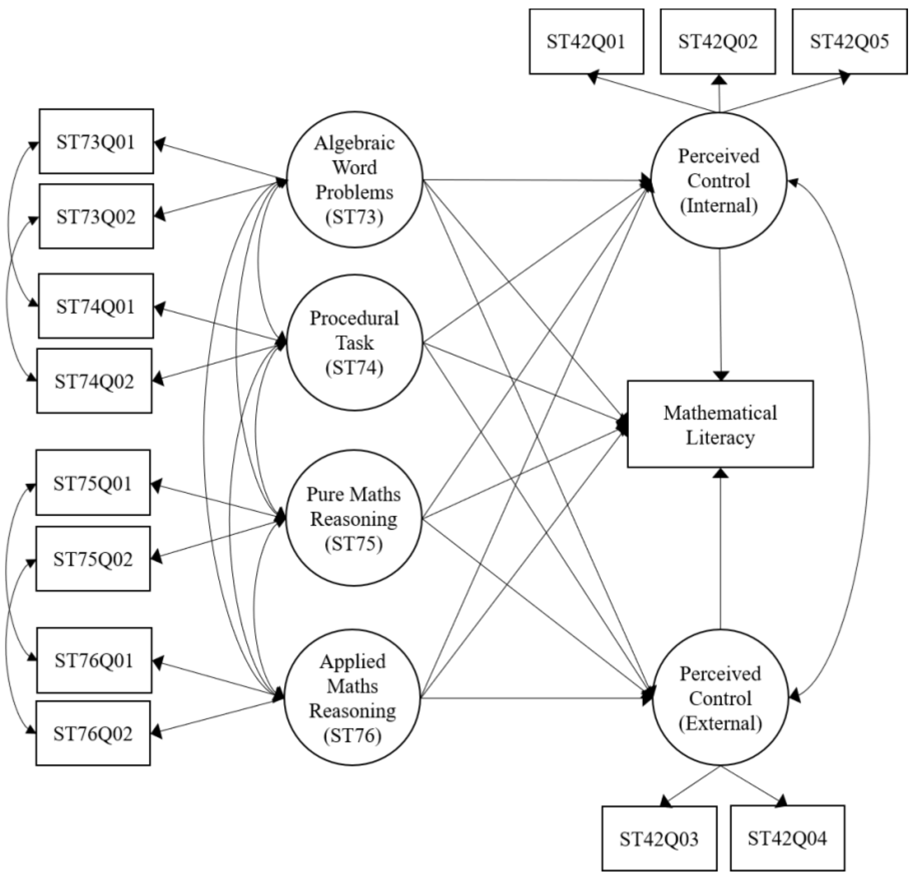


Figure 1. Full Model of the Relationship among OTL, Perceived Control, and Mathematical Literacy

It should be noted that each of the latent variables of OTL and EPC was estimated using two indicators due to the structure of the PISA 2012 data. It is commonly recommended to have more than two indicators per latent variable. However, some researchers argued that one or two indicators could be sufficient (Hayduk & Littvay, 2012). Furthermore, the analysis did not yield any errors that are very likely to happen with two indicators (e.g., negative residual variances known as Haywood cases), and the number of indicators showed little effect of bias (Little et al., 1999). For those reasons, the estimated model is valid to interpret the relationship between OTL and mathematical literacy.

Thus, we are interested in both direct relationship and indirect relationships through perceived control, between mathematical tasks and mathematical literacy. We attempted to test the positive relationships between OTL with different tasks and mathematical literacy by examining the hypothesized model with the structural equation modeling. When students learn mathematics through OTL with mathematical tasks, the positive relationships between the tasks and mathematical literacy are somewhat expected. Particularly, we expected that students' opportunity to engage in applied mathematics reasoning tasks would be more strongly related to mathematical literacy based on the definition of mathematical literacy given by the PISA 2015.

METHOD

Participants

We utilized the PISA 2012 international database, which is open to the public. The rationale to use this database instead of the PISA 2015 or 2018 was that the focus subjects of these recent PISA studies were not mathematics. The variables included to address our research questions were available only in the PISA 2012. Among 5,033 Korean students in the original PISA 2012 database we collected responses of 1,649 Korean students who participated in both student questionnaire and mathematical literacy assessment. The PISA 2012 student context questionnaires in a rotation design, which consisted of the 'common' question (answered by all students) and 'rotated' questions (answered by two thirds of the student sample; OECD, 2014, p. 59). Because all of the survey items used in this research (ST43, questions asking 'Thinking about your mathematics lessons: to what extent do you agree with the following statements?' and ST73-76, mathematical tasks) were included together in the form A, the rotation questionnaires design allowed us to observe only students taking this form, a third of the Korean students participating in the PISA 2012 (See figure 3.9 in OECD, 2014, p. 61).

Variables

Mathematical literacy. As seen in the hypothesized model, we collected all sets of plausible values representing students' mathematical literacy scores provided in the PISA 2012. Large-scale international studies such as TIMSS and PISA do not provide one value for each student's achievement in mathematics. Rather, as Foy, Brossman, and Galia (2012) argued, plausible values are provided through the process called "conditioning" with all background variables, for which relationships between background variables and mathematics achievement can serve as a satisfactory explanation.

Table 1. Weighted Means and Standard Deviations of Mathematical Literacy

Plausible Value	PV1	PV2	PV3	PV4	PV5
Weighted Mean	553.57	553.53	555.08	553.56	553.44
Standard Deviation	99.61	100.22	99.53	100.52	101.30

Furthermore, we highlighted that "[p]lausible values are not test scores and should not be treated as such" (OECD, 2014, p. 147) and the plausible values should be analyzed in a correct way. According to von Davier, Gonzalez, and Mislevy (2009), averaging plausible values themselves to have one value representing students' mathematical literacy could lead to biased estimates. Chaney et al. (2001) suggested conducting separate analysis with each set of plausible values and average the estimated parameters. We also applied some formulas that Chaney and his colleagues provided to compute the standard errors for calculated estimates. Lastly, mathematical literacy was standardized in the SEM analysis. Table 1 shows the weighted mean and the standard deviation of each set of the plausible values.

Perceived control. We collected students' responses to the question, given the code, ST43, asking students' degrees of agreements to six statements in Table 2. For data analysis, we assigned "4" to students' strong agreement to each statement, "3" to moderate agreement, "2" to moderate disagreement, and "1" to strong disagreement. Though the way of assigning numbers to students' responses is different from the way used in the PISA 2012, our method allows us to interpret that higher numbers of students' responses indicate stronger agreement to the statements about perceived control.

After selecting the data of the question ST43, we categorized the six statements into the two: internal (IPC) and external perceived control (EPC) based on the discussion to build the hypothesized model. Internal perceived control was measured through the three statements – ST43Q01, ST43Q02, and ST43Q05. Other two statements – ST43Q03 and ST43Q04 – were used to measure external perceived control. We excluded the statement, ST43Q06 that asked about test preparation because it is neither internal nor external perceived control. This statement can imply that students' performance on mathematics exams is irrelevant to their efforts, but the statement itself does not allows us to identify the statement as either internal or external perceived control.

As seen in Table 2, more than 85% of Korean students agreed with the three statements ST43Q01, ST43Q02, and ST43Q05, which were used to measure internal perceived control. Simultaneously, most Korean students disagreed with the other statements about external perceived control and test preparation. When ST43Q03 and ST43Q04 were compared, it was interesting that more students strongly agreed that their success/failure is attributed to their teachers.

Table 2. The Number of Students and Weighted Percentage by Response to the Six Statements

Code	**Question:** Thinking about your mathematics lessons: to what extent do you agree with the following statements? Statement	Frequency			
		Strongly Agree (4)	Agree (3)	Disagree (2)	Strongly Disagree (1)
ST43Q01	If I put in enough effort I can succeed in mathematics.	503 (30.1%)	945 (57.7%)	166 (10.1%)	35 (2.1%)
ST43Q02	Whether or not I do well in mathematics is completely up to me.	612 (37.0%)	912 (55.5%)	93 (5.6%)	32 (1.9%)
ST43Q03	Family demands or other problems prevent me from putting a lot of time into my mathematics work.	66 (3.9%)	311 (18.8%)	940 (57.2%)	332 (20.1%)
ST43Q04	If I had different teachers, I would try harder in mathematics.	116 (7.1%)	333 (20.0%)	912 (55.3%)	288 (17.6%)
ST43Q05	If I wanted to, I could do well in mathematics.	482 (29.0%)	951 (57.9%)	172 (10.5%)	44 (2.6%)
ST43Q06	I do badly in mathematics whether or not I study for my exams.	108 (6.6%)	425 (26.0%)	793 (48.2%)	323 (19.2%)

Opportunity to Learn. Students' OTL data were collected with the responses to four questions (ST73, ST74, ST75, and ST76). Each question included two sub-questions showing different types of tasks: "how often have you encountered these types of problems in your mathematics lessons (ST[73–76]Q01)?; and "in the tests you have taken at school (ST[73–76]Q02)?" We highlight that the four questions focused on students' perception of *how often* they encountered OTL with the tasks, among other dimensions of OTL (Stevens & Grymes, 1993). Table 3 shows the detailed questions for OTLs labeled with *algebraic word problem* (WP; ST73) and *procedural tasks* (PT; ST74) in the PISA 2012. Table 4 also shows the other two questions used to identify OTL with *pure mathematics reasoning* (PMR) and *applied mathematics reasoning* (AMR).

Table 3. OTL Questions for Algebraic Word Problem and Procedural Task (OECD, n.d.)

Algebraic Word Problem (WP; ST73)	**Question.** In the box is a series of problems. Each requires you to understand a problem written in text and perform the appropriate calculations. Usually the problem talks about practical situations, but the numbers and people and places mentioned are made up. All the information you need is given. Here are two examples: **Example 1)** <Ann> is two years older than <Betty> and <Betty> is four times as old as <Sam>. When <Betty> is 30, how old is <Sam>? **Example 2)** Mr. <Smith> bought a television and a bed. The television cost <$625> but he got a 10% discount. The bed cost <$200>. He paid <$20> for delivery. How much money did Mr. <Smith> spend?				
	# of Students	Frequently (4)	Sometimes (3)	Rarely (2)	Never (1)
	ST73Q01	545 (33.1%)	821 (49.8%)	215 (13.1%)	68 (4.0%)
	ST73Q02	336 (20.5%)	792 (48.2%)	410 (24.7%)	111 (6.6%)

Procedural Tasks (PT; ST74)

Question. Below are examples of another set of mathematical skills.
Example 1) Solve 2x + 3 = 7.
Example 2) Find the volume of a box with sides 3m, 4m and 5m.

	# of Students	Frequently (4)	Sometimes (3)	Rarely (2)	Never (1)
Procedural Tasks (PT; ST74)	ST74Q01	968 (58.9%)	540 (32.6%)	109 (6.6%)	32 (1.8%)
	ST74Q02	746 (45.5%)	623 (37.8%)	217 (13.1%)	63 (3.7%)

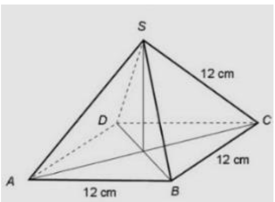
Note. All percentages were weighted.

An interesting finding from Tables 3 and 4 is that more than 70% of students answered that they encountered each of WP, PT, and PMR frequently in their mathematics lessons and tests. However, approximately a half of students reported that they encountered AMR frequently or sometimes. Particularly, 91.4% of students encountered PT at least sometimes, whereas 44.2% of students saw AMR tasks at most rarely in lessons. This indicates that there were substantial gaps in the frequencies of OTL with different tasks that were offered to Korean students; specifically, limited OTL with AMR that requires students to make sense of real problem situations and interpret/explain the solutions. This is a unique mathematical process that AMR offers, while other task types do not.

Table 4. OTL Questions for Pure Mathematics Reasoning and Applied Mathematics Reasoning (OECD, n.d.)

Pure Mathematics Reasoning (PMR; ST75)

Question. In the next type of problem, you have to use mathematical knowledge and draw conclusions. There is no practical application provided. Here are two examples.

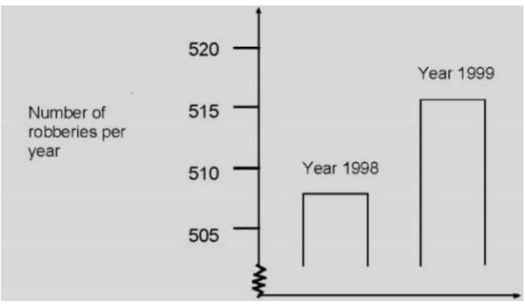


Example 1) Here you need to use geometrical theorems: Determine the height of the pyramid.
Example 2) If n is any number: can $(n+1)^2$ be a prime number?

	# of Students	Frequently (4)	Sometimes (3)	Rarely (2)	Never (1)
Pure Mathematics Reasoning (PMR; ST75)	ST75Q01	560 (34.0%)	722 (44.1%)	281 (16.8%)	86 (5.0%)
	ST75Q02	511 (31.0%)	719 (44.0%)	308 (18.6%)	111 (6.5%)

Applied Mathematics Reasoning (AMR; ST76)

Question. In this type of problem, you have to apply suitable mathematical knowledge to find a useful answer to a problem that arises in everyday life or work. The data and information are about real situations. Here are two examples.
Example 1) A TV reporter says "This graph shows that there is a huge increase in the number of robberies from 1998 to 1999."



Do you consider the reporter's statement to be a reasonable interpretation of the graph? Give an explanation to support your answer.
Example 2) For years the relationship between a person's recommended maximum heart rate and the person's age was described by the following formula:

Recommended maximum heart rate = 220 – age

	Recent research showed that this formula should be modified slightly. The new formula is as follows: *Recommended maximum heart rate = 208 – (0.7 × age)* From which age onwards does the recommended maximum heart rate increase as a result of the introduction of the new formula? Show your work.			
# of Students	Frequently (4)	Sometimes (3)	Rarely (2)	Never (1)
ST76Q01	203 (12.4%)	717 (43.6%)	564 (34.1%)	165 (9.8%)
ST76Q02	187 (11.3%)	631 (38.6%)	618 (37.2%)	218 (12.9%)

Note. All percentages were weighted.

Data Analysis

Using the variables described above, we applied the structural equation modeling (SEM) to evaluate the hypothesized model in Figure 1. The SEM approach was utilized with the R package *lavaan.survey* (Obserski, 2016) and maximum likelihood estimation that considered all variables as continuous. The strength of this R package was that the complex PISA 2012 hierarchical design could be fully considered in the SEM analysis using students' weights and balanced repeated replications (BRR). First, because our research interests were solely at the student level, the data analysis required to use student weights in data analysis (Asparouhov & Muthen, 2006). The PISA 2012 provided "final trimmed nonresponse adjusted student weight," which was calculated with the consideration of stratified sampling design. Thus, statistical results such as descriptive statistics and SEM results were weighted. Second, weighting was not enough to make unbiased decisions when multilevel sampling was applied. Particularly, "the variance estimator can be unstable" relying on the sample design (OECD, 2017, p. 123). To resolve this issue, it was recommended to use BRR to estimate sampling variances (OECD, 2017). In this research, we employed Fay's method of BRR by using variables "final student replicate BRR-Fay weights" in the databases.

Considering the complexity of the analysis using the five sets of plausible values and Fay's method of BRR, we applied the three steps of the SEM approach suggested by Byrne (1998): model specification, model assessment, and model respecification. First, as discussed in the previous section, the hypothesized model in Figure 1 was already constructed based on the relevant literatures. Second, the followings were evaluated for the next stage of respecification: the overall model fits, the suitability of parameter estimates, and the statistical significance of parameter estimates. It was checked that the outputs included some error messages like negative variances, correlations greater than 1, and non-positive definite covariance matrices, which all are unreasonable (Bryne, 1998). Furthermore, non-significant parameters could slightly contribute to the power of the model to explain the phenomenon. Thus, we considered the parameters with $p < 0.1$ because we had less concern of Type I error. For the overall model evaluation, all model fits were comprehensively evaluated using criteria summarized by Schreiber, Nora, Stage, Barlow, and King (2006, p. 330) – the comparative fit index (CFI), the Tucker-

Lewis fit index (TLI), the standardized root mean square residual (SRMR), and the root mean square error of approximation (RMSEA).

On one hand, we removed some non-significant indicators to respecify the hypothesis model after model assessments. On the other hand, we included correlated residuals having large modification indices representing expected changes in model fits. However, we should have theoretical rationales in addition to the statistical evidence to add correlated residuals (Kline, 2011). Accordingly, we considered four pairs of correlated residuals: (1) ST73Q1 and ST74Q1, (2) ST73Q2 and ST74Q2, (3) ST75Q1 and ST76Q1, and (4) ST75Q2 and ST76Q2. These pairs showed statistical evidence of large modification indices. Also, it is noticeable that residuals of the questions about WP and PT, and PMR and AMR were correlated, which could be due to the cognitive demands of mathematics tasks (Boston & Smith, 2009).

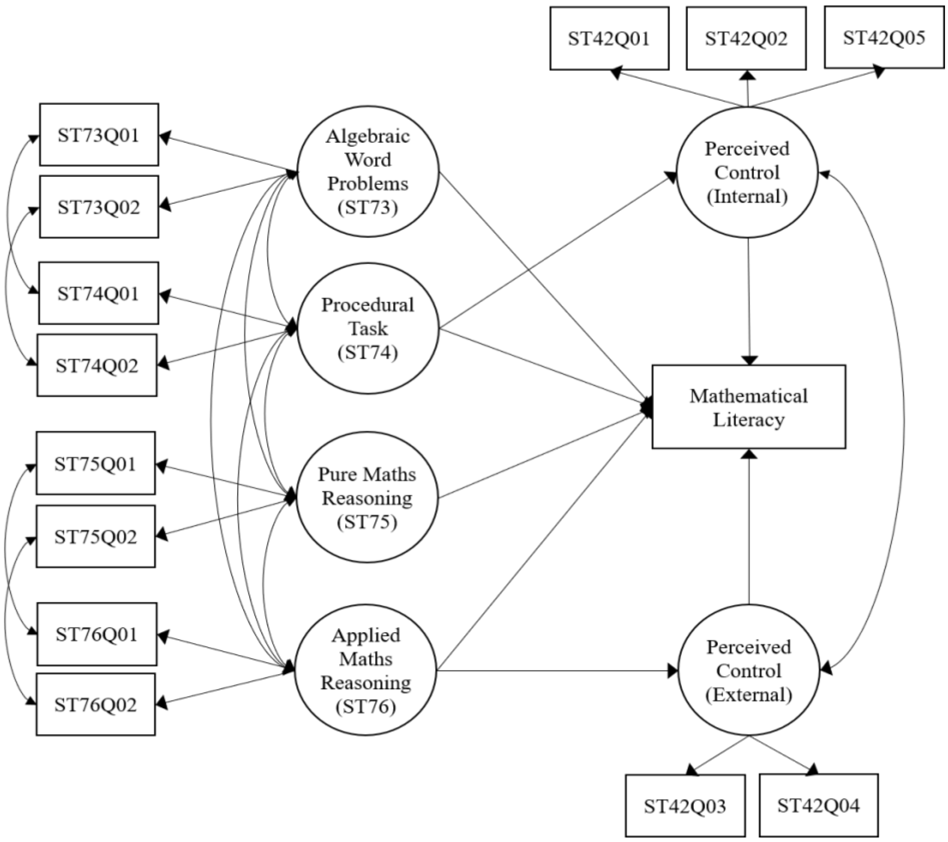


Figure 2. Nested Model Respecified from the Full Model

After the respecification, we compared the full model (see Figure 1) and the nested model (see Figure 2). Using χ^2 tests, AIC, BIC, and sample-size adjusted BIC, the comparison could answer whether there was a significant difference in the goodness of fit between the nested and full models, though the nested model had a smaller number of parameters. If we retain the null hypothesis that there is no significant difference between the models, we prefer the nested model to the full model, because the nest model had a similar goodness of fit with less parameters. Then, the SEM results of the relationships among OTLs, perceived control, and mathematical literacy would be based on the regression weights in the selected model.

We highlight that estimation methods in SEM (e.g., maximum likelihood in this study) require a normality assumption of endogenous variables. However, it is known that parameter estimates are robust against violation of normality assumption while Type I error rate of hypothesis tests on individual parameters are likely to be inflated. Furthermore, plausible values of mathematical literacy measured in PISA 2015 were constructed based on normal population distributions. Thus, we argued the robustness of the model.

RESULTS AND DISCUSSION

We will report the fit indices to compare the nested and full models to answer our research questions. After the discussion of model selection process, we will report the SEM results to explain the relationships among OTLs, perceived control, and mathematical literacy.

Table 5. Model Fit Indices using Five Plausible Values

		CFI	TLI	Information Index			RMSEA	SRMR	χ^2 test
				AIC	BIC	Adjusted BIC	Point Estimate & 90% Confidence Interval		*p*-value
PV1	Full	0.977	0.962	43687.9	44036.9	43830.4	0.049 (0.043 0.055)	**0.021**	0.784
	Nested	0.977	**0.966**	**43679.6**	**43996.3**	**43808.9**	**0.046 (0.040 0.052)**	0.022	
PV2	Full	0.977	0.961	43715.9	44064.9	43858.4	0.049 (0.043 0.055)	**0.022**	0.785
	Nested	0.977	**0.965**	**43707.5**	**44024.3**	**43836.8**	**0.046 (0.041 0.052)**	0.023	
PV3	Full	0.976	0.960	43678.0	44027.0	43820.5	0.050 (0.044 0.056)	0.023	0.780
	Nested	**0.977**	**0.964**	**43669.7**	**43986.4**	**43799.0**	**0.047 (0.041 0.053)**	0.023	
PV4	Full	0.978	0.962	43702.3	44051.3	43844.8	0.048 (0.042 0.054)	**0.022**	0.784
	Nested	0.978	**0.966**	**43693.9**	**44010.7**	**43823.3**	**0.046 (0.040 0.051)**	0.023	
PV5	Full	0.977	0.961	43702.6	44051.6	43845.1	0.049 (0.043 0.055)	**0.022**	0.787
	Nested	0.977	**0.966**	**43694.2**	**44011.0**	**43823.5**	**0.046 (0.040 0.052)**	0.023	

Note. Bold numbers indicate the better model between the nested and full models.

Model Comparison

The model fit indices of the full and nested models were estimated across all sets of plausible values of mathematical literacy. Based on the recent criteria (CFI ≥ 0.95, TLI ≥ 0.95, RMSEA < 0.06 with confidence interval, and SRMR ≤ 0.08; Schreiber et al., 2006), all indices of both models were acceptable. When we compared the full and nested models, the χ^2-test results indicated that there were no significant differences in the goodness of fit between the two models as seen in the last column of Table 5. Additionally, most indices – CFI, TLI, RMSEA – showed that the nested model had slightly better fit indices with the smaller number of the parameters. Less values of information indices (AIC, BIC, and adjusted BIC) indicated a better model, which led to the same conclusion. Thus, we selected the nest model in Figure 2, which was a more parsimonious model with the similar goodness of fit.

SEM Results

Table 6 reports the estimated measurement model in the standardized metric. Lower factor loadings indicate that the corresponding indicators were conceptually distant from the latent variables. All factor loadings except for that of ST43Q03 were greater than 0.4 with $p < 0.001$, which satisfied previously suggested recommendations (Tabachnick & Fidell, 2007). Although the factor leading of ST43Q03 was 0.387, we argue that this coefficient was acceptable. However, the factor loadings for external perceived control were similar, which means that both statements can reflect conceptually similar distance of different facets of external perceived control – teachers and family.

Table 6. Results from the Measurement Model

Observed Variable	Latent Variables	Coefficient	SE	z-value	*p*-value
ST73Q01	WP	0.675	0.026	25.502	<0.001
ST73Q02	WP	0.621	0.022	27.759	<0.001
ST74Q01	PT	0.632	0.023	27.447	<0.001
ST74Q02	PT	0.642	0.022	28.933	<0.001
ST75Q01	PMR	0.786	0.022	35.501	<0.001
ST75Q02	PMR	0.751	0.022	33.926	<0.001
ST76Q01	AMR	0.746	0.021	35.454	<0.001
ST76Q02	AMR	0.744	0.021	35.330	<0.001
ST43Q01	IPC	0.540	0.018	29.934	<0.001
ST43Q02	IPC	0.444	0.019	23.347	<0.001
ST43Q05	IPC	0.505	0.017	29.676	<0.001
ST43Q03	EPC	0.387	0.056	6.910	<0.001
ST43Q04	EPC	0.439	0.069	6.332	<0.001

The SEM results included the correlation coefficients between OTLs with different types of tasks as seen in Table 7. Overall, all OTLs were highly correlated with each other, which means that if students have more frequent OTLs with a certain type of task, they were very likely to do so with others. It should be noted that the correlation coefficient between AMR and PT was relatively low, 0.258. This correlation indicates that AMR with PT was somewhat independent compared to other pairs of OTLs.

Table 7. Correlation Coefficients between Latent Variables

	WP	PT	PMR	AMR
WP	1	0.572	0.387	0.396
PT		1	0.426	**0.258**
PMR			1	0.428
AMR				1

Note. All correlation coefficients are significant with $p < 0.01$

Figure 3 shows all SEM results based on the nested model. Table 8 reports the regression weights, which were all significant at the alpha level of 0.05. First, only the regression weight of AMR was negative (-0.174), the others were positive (0.132 for WP, 0.099 for PT, and 0.104 for PMR). The degree of this negative effects was also larger than others. IPC has much stronger relationship to mathematical literacy than EPC. It is remarkable that EPC is negatively related to mathematical literacy although it is not statistically significant at alpha 0.05. In addition, students' EPC was expected to increase by 0.091 when students had increase of 1 in their AMR. These findings about AMR indicate that encountering AMR frequently had negative influences on their mathematical literacy and positive influences on EPC simultaneously. However, OTL with PT was expected to increase mathematical literacy scores both directly and indirectly through IPC. When students increased 1 in their PT, students were expected to have increase in their IPC by 0.297. This value was remarkably large when considering that IPC was a psychological factor and that this result indicated possible impact of tasks introduced to students on their psychological perception. Because students' increase in IPC by 1 was expected to increase mathematical literacy by 0.358, the indirect effect of PK was 0.106 = 0.297×0.358 (SE = 0.016, $p <$ 0.001). This value was similar with the direct effect of PT, 0.099 ($p = 0.004$).

Table 8. Results from the Structural Equation Modeling

Independent Variable	Dependent Variable	Coefficient	SE	z-value	p-value
PT	IPC	0.297	0.039	7.604	<0.001
AMR	EPC	0.091	0.043	2.120	0.034
IPC	Mathematical Literacy	0.358	0.030	11.970	< 0.001
WP	Mathematical Literacy	0.132	0.035	3.738	< 0.001
PT	Mathematical Literacy	0.099	0.044	2.259	0.024
PMR	Mathematical Literacy	0.104	0.036	2.914	0.004
AMR	Mathematical Literacy	-0.174	0.032	-5.386	< 0.001

This research studied the relationship among OTL, perceived control, and mathematical literacy. OTL itself is conceptualized as the frequency of engagement in four different mathematical tasks that was perceived by students. It is critical to think about the possible explanations for this finding, though verifying the speculated reasons is beyond the scope of this study. Here, we provide possible explanations that may be implemented by subsequent studies and, by extension, identify pathways for future research. First, we investigated the relationship between mathematical tasks as OTL and two types of perceived control: internal and external. On the one hand, students' OTL with procedural tasks was positively related to internal perceived control ($p < 0.001$); on the other hand, OTL with applied mathematics reasoning tasks was related to external perceived control ($p = 0.034$). Although these significant relationships were not necessarily causal, this suggests that students' experiences with different types of tasks in mathematics classrooms are one of the factors that shape students' perceptions of perceived locus of control.

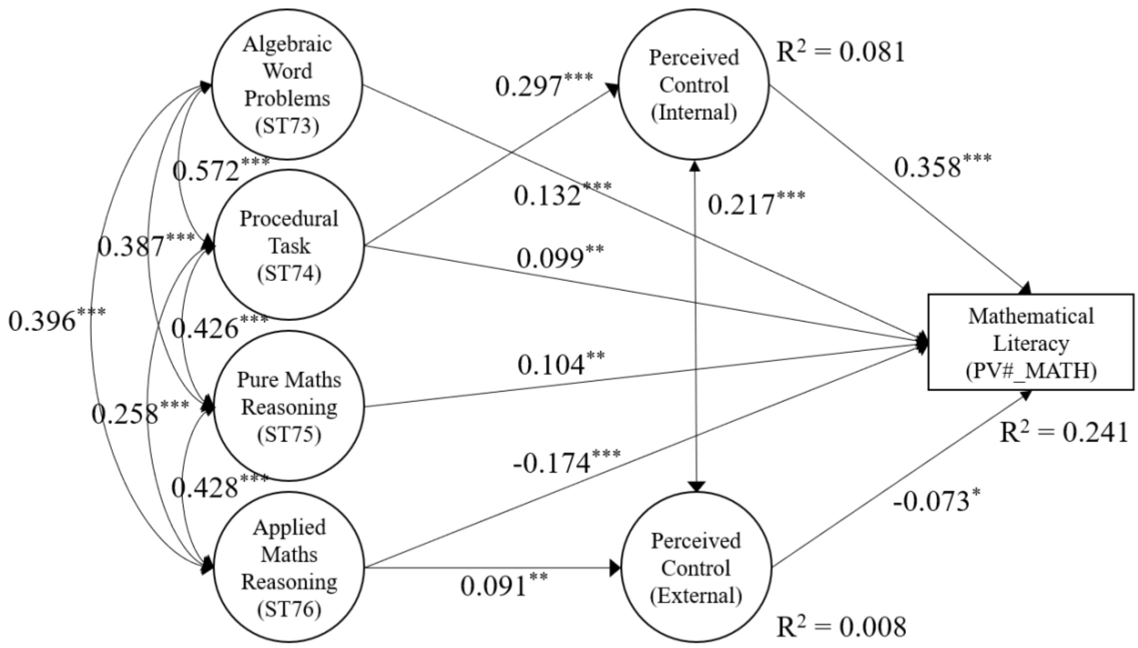


Figure 3. The SEM Results based on the Nested Model

When speculating several reasons for why perceived control and OTL diverged as observed, we put forward how students' engagement in mathematical tasks can relate to variations in their teachers' implementation of the tasks. Specifically, there could be larger variances in ways to implement of applied reasoning tasks (involving high cognitive demands) than in procedural tasks (involving low cognitive demands). In this sense, students' OTL in applied mathematical reasoning tasks can also vary according to how teachers' implement those tasks, which includes how the tasks are presented and how students' learning is scaffolded by teachers. Since cognitive demand for tasks varies by the ways of teachers' task presentation and scaffolding, students might perceive that their success/failure of the learning tasks is subject to teachers' implementation and that their success/failure is out of their control. In contrast to the applied reasoning tasks, procedural tasks do not give as much space for implementation-variation, and hence, students' positive/negative experiences with procedural tasks are perceived not to be contingent on how they are implemented by teachers. In this study, students who were frequently engaged in procedural tasks were more likely to think that their success/failure is under their own control, which resulted in a strongly positive relationship between OTL in procedural tasks and internal perceived control.

In this study, we also investigated the relationship between OTL via mathematical tasks and mathematical literacy scores. In constructing the model in Figure 1, we conjectured that all types of OTL would be positively related to mathematical literacy, though the degree of the relationship may vary. Particularly, we expected that applied mathematical reasoning tasks would have a stronger positive relationship to mathematical literacy than the other three task types. The PISA assessment for mathematical literacy involved items that assessed capabilities in mathematical reasoning, and use of mathematical concepts and procedures in various real contexts. At the beginning of the study, we reasoned that many constituent capacities of mathematical literacy are utilized when students engage in applied mathematical reasoning tasks; however, the findings challenged this conjecture. Indeed, OTL via applied mathematical reasoning was negatively related to mathematical literacy although all other mathematical tasks (word problems, procedural tasks, and pure mathematics reasoning tasks) were positively related to mathematical literacy. This means that those students who were exposed more frequently to applied mathematical reasoning tasks were likely to have lower mathematical literacy scores, whereas those who were more frequently engaged in other types of tasks were likely to have higher mathematical literacy scores.

At this point, we will discuss why OTL via applied reasoning mathematical tasks had a uniquely different relationship with mathematical literacy compared to the other three tasks. We will highlight the specific cognitive processes that are required to successfully engage in each type of task, and how we can characterize such thinking processes. According to Hanna and Jahnke (2007), engagement in mathematical tasks requires students to undergo two reasoning processes: (1) making abductive inferences, such as "an action of *selection,*" to build the correct premise and (2) making deductive inferences between the premise and the conclusion (p. 149). As presented in the example tasks (Tables 3 and 4), pure mathematical reasoning, word problems, and procedural tasks offer the correct premises directly to the students, thereby placing less emphasis on abductive inferences. For applied mathematical reasoning, abductive inference is one of the most important elements when formulating a given real situation and building premises.

Also, theoretically, mathematical literacy is defined as a combination of both abductive and deductive inferences, which includes the abilities of formulating premises, employing mathematical concepts and ideas, and interpreting solutions in real situations. However, the mathematical literacy that was measured in the PISA 2012 might not have captured both abductive and deductive inferences in a balanced way. Even though the OECD reported that PISA mathematics assessments are improved by using computer-based delivery formats, it is still difficult to evaluate students' inductive/abductive reasoning skills with mathematics test items. Students are asked to answer multiple-choices questions in the assessments, and this type of assessment does not well reflect students' processes of searching for reasons. Thus, in standardized test settings, often with time limits, students focus more on finding a correct answer from the information that is presented in the test problems, rather than exploring and formulating a real-word problem situation.

The questionnaire in PISA 2012 asked students how often they encountered each type of tasks during their mathematics classes. The negative relationship between applied reasoning mathematical tasks and mathematical literacy encouraged us to rethink how the frequency of OTL via applied reasoning tasks can affect mathematics learning. Departing from the idea of 'the more, the better,' we speculated that the implementation of applied reasoning tasks has much to do with how they are implemented, as opposed to how often they are implemented. Considering that the PISA student survey was about how often students encountered each type of tasks, it is possible that other important aspects of OTL, such as quality and process of OTL, were not taken into consideration. The varying quality of students' OTL of applied mathematical reasoning could be another reason for the negative relationship between applied mathematics reasoning tasks and mathematical literacy. Particularly, some researchers (e.g., Boston & Smith, 2009) have argued that teachers tend to reduce the original cognitive demands of mathematical tasks when implementing them. This means that the result is probably due to the ways in which those reasoning tasks were implemented. It can be challenging for teachers to scaffold students' learning process carefully and successfully by engaging them in applied mathematics reasoning tasks. As such, this may inhibit students from fully taking advantage of OTL via applied mathematical reasoning.

The findings of this study support that allocating more learning time to applied reasoning task is not necessarily beneficial to, or does not guarantee, overall mathematics learning. However, our attempts made so far to explain the *negative* relation between applied reasoning tasks and mathematical literacy still may not seem to be sufficient. Thus, future research on teachers' implementation of applied mathematics reasoning tasks in classrooms should be followed to validate and explain the negative relationship between applied mathematics reasoning task and mathematical literacy.

Another research question of our study was on the role of perceived control in the relationship between OTL with mathematical tasks and mathematical literacy. The results showed that engagement in OTL with procedural tasks is likely to influence mathematical literacy directly and indirectly through internal perceived control. Particularly, the effect of engagement in procedural tasks on mathematical literacy is even greater when the indirect effect through internal perceived control is taken into consideration. Considering the strong positive relationship between internal perceived control and mathematical literacy, students are likely to have high mathematical literacy scores when they believe that being successful in mathematics is under their control. To synthesize, OTL through procedural tasks is likely to promote students' internal perceived control, and in turn, this can have an effect on better mathematical literacy. Though this may suggest the merit of engaging students in procedural tasks in relation with students' perceived control, we are not to argue that procedural tasks should be offered more in mathematics classrooms than other types of tasks. As Yeo (2007) argued, students need to have a variety of OTL, from procedural tasks to mathematizing tasks, and teachers need to be cognizant about different OTL that is afforded by various types of tasks. This is specifically because OTL through different types of tasks may have varying effect on cognitive and non-cognitive processes during mathematics learning, as shown in our study.

CONCLUSION

This research showed that students can improve their mathematical literacy by engaging in various types of tasks from procedural tasks, word problems, to pure and applied mathematics reasoning tasks. Opposite from our expectation, the results showed that students are likely to have lower mathematical literacy when they have encountered applied mathematics reasoning tasks more frequently. In addition to the discussion of the results, we suggest future studies about how different types of tasks are implemented in classrooms, how such implementation influence students' perceived control, and how students perform on tests based on their classroom experiences.

We suggest several implications based on the findings in this research: First, teachers and curricular developers need to implement various tasks considering their relation to students' perceived control and mathematical literacy. However, it is important to recognize that students' frequent engagement in certain tasks could have unexpected influences on their mathematical literacy, especially when they are not appropriately facilitated. Particularly, Korean students' OTL with applied mathematics reasoning tasks had negative relationship with their achievement. These findings call for more investigation on how to implement such tasks appropriately.

Secondly, it is critical to consider how teachers select OTL with different types of tasks to offer in their teaching practices. When educators emphasize tasks with high cognitive demands, it is sometimes misunderstood that tasks with low cognitive demand are less beneficial to students' higher order thinking processes in mathematics learning. Even worse, tasks with low cognitive demand are considered as something that teachers should avoid. However, our findings show that procedure tasks can help students believe that their success is under their own control, which could lead to better mathematics learning behaviors and higher mathematical literacy through appropriate scaffoldings for other types and levels of mathematical tasks. Then, students will be able to engage in different types of mathematical thinking and perceive that they can succeed in mathematics learning with their own effort.

We did not make direct relation between educational contexts of Korea and the results of this study in our interpretation of the results, which should be noted when attempting to generalize the findings to different educational systems. Moreover, we highlight that mathematical literacy defined and measured by the PISA could be different from achievement measures in other mathematics assessments. Therefore, replication studies using other assessments tools or in other educational contexts are needed.

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INDONESIAN MATHEMATICS TEACHERS' KNOWLEDGE OF CONTENT AND STUDENTS OF AREA AND PERIMETER OF RECTANGLE

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Abstract

Measuring teachers' skills and competencies is necessary to ensure teacher quality and contribute to education quality. Research has shown teachers competencies and skills influence students' performances. Previous studies explored teachers' knowledge through testing. Teachers' knowledge of the topic of area-perimeter and teaching strategies has been assessed through testing. In general, items or tasks to assess mathematics teacher knowledge in the previous studies were dominated by subject matter knowledge problems. Thus, it seems that the assessment has not fully covered the full range of teacher knowledge and competencies. In this study, the researchers investigated mathematics teachers' Knowledge of Content and Students (KCS) through lesson plans developed by the teachers. To accommodate the gap in the previous studies, this study focuses on KCS on the topic of area-perimeter through their designed lesson plans. Twenty-nine mathematics teachers attended a professional development activity voluntarily participated in this study. Two teachers were selected to be the focus of this case study. Content analysis of the lesson plan and semi-structured interviews were conducted, and then data were analyzed. It revealed that the participating teachers were challenged when making predictions of students' possible responses. They seemed unaware of the ordinary students' strategies used to solve maximizing area from a given perimeter. With limited knowledge of students' possible strategies and mistakes, these teachers were poorly prepared to support student learning.

Keywords: Knowledge of Content and Students, Mathematics Teacher, Area and Perimeter, Teachers' Skills and Competencies

Abstrak

Mengukur keterampilan dan kompetensi guru diperlukan untuk memastikan kualitas guru dan berkontribusi pada kualitas pendidikan. Penelitian ini menunjukkan bahwa komptensi dan keterampilan guru mempengaruhi performa siswa. Penelitian sebelumnya telah mengkaji pengetahuan gru melalui tes. Pengetahuan guru pada topik keliling-luas dan strategi pembelajaran juga telah dikaji melalui tes. Pada umumnya, banyaknya soal pada tes didominasi oleh soal-soal tentang pengetahuan subjek yang diajarkan. Oleh karena itu, asesmen seperti ini belum mencakup kesuluruhan pengetahuan dan kompetensi guru. Pada studi ini, peneliti menginvestigasi pengetahuan guru matematika tentang KCS pada rencana pelaksanaan pembelajaran yang mereka kembangkan. Untuk mengakomodasi kesenjangan pada penelitian sebelumnya, penelitian kali ini berfokus pada pengetahuan tentang konten dan siswa (KCS) pada topik keliling-luas pada rencana pelaksanaan pembelajaran. Dua puluh Sembilan guru matematika yang sedang mengikuti pelatihan peningkatan kompetensi secara suka rela mengikuti penelitian ini. Dua guru matematika menjadi fokus penelitian studi kasus ini. Konten analisis dan interview semi terstruktur dilakukan dan datanya dianalisis. Terungkap bahwa peserta ini mengalami tantangan dalam memprediksi kemungkinan respon yang diberikan siswa. Mereka belum menyadari strategi siswa yang biasanya digunakan untuk menyelesaikan persoalan memaksimalkan luas dari keliling yang ditentukan. Dengan pengetahuan yang terbatas pada kemungkinan strategi siswa dan kesalahan siswa, guru ini kurang siap dalam mendukung siswanya

Kata kunci: Pengetahuan tentang Materi dan Siswa, Guru Matematika, Luas dan Keliling, Keterampilan dan Kompetensi Guru

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Shulman (1986) refers to Pedagogical Content Knowledge (PCK) as the ways of representing and formulating the subject that is understandable to others. Research have shown that student achievements are more affected by PCK than Subject Matter Knowledge (SMK) as the quality of instruction is related to PCK (Baumert et al., 2010; Hill, Rowan, & Ball, 2005; Hill, Ball, & Schilling, 2008). As the use of SMK terminology varies, SMK in this paper refers to common content knowledge (CCK) which is part of SMK (see Figure 1).

> *Hill, Ball and Shilling (2008), in seeking to conceptualize the domain of effective teachers' unique knowledge of students' mathematical ideas and thinking, proposed the following domain map for mathematical knowledge for teaching (see Figure 1) (White, et al., 2012, p.394).*

One specific aspect of PCK is the Knowledge of Content and Students (KCS). KCS is ‘knowledge that combines knowing about students and knowing about mathematics (Ball, Thames, & Phelps, 2008, p. 401). It consists of anticipating what students are likely to think about, what they could find confusing or complicated, and what students are expected to do mathematically to complete the chosen task.

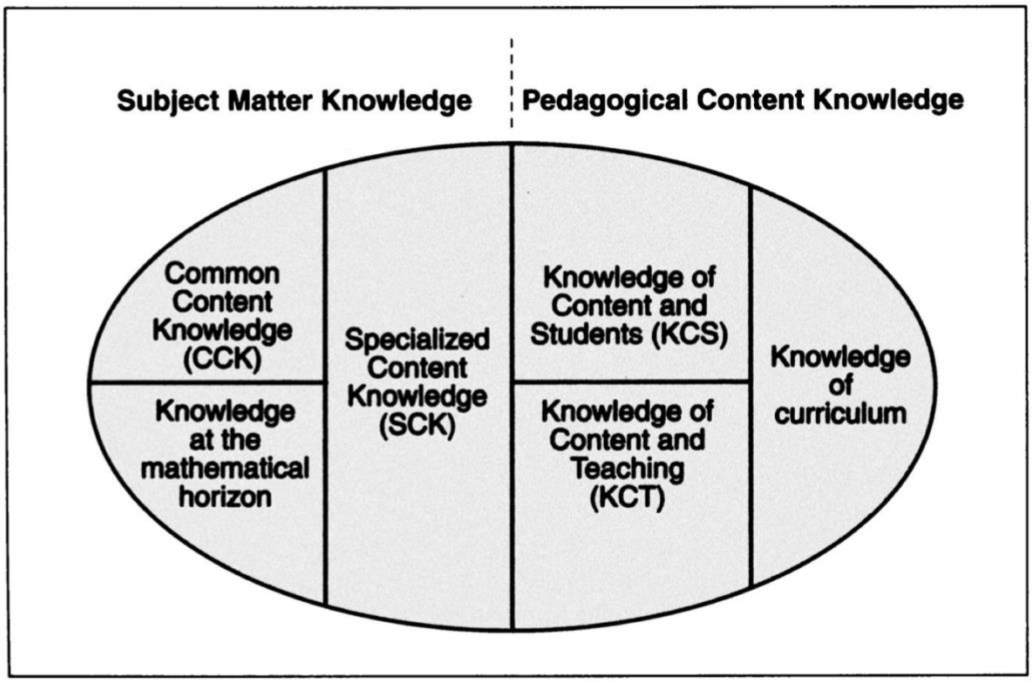

Figure 1. Domain map for mathematical knowledge for teaching (Hill, Ball, & Schilling, 2008, p. 377)

There are some teacher assessment models which measure knowledge for teaching. The Teacher Education and Development Study in Mathematics (TEDS-M) is one of the international assessments intended for pre-service mathematics teachers (Tatto et al., 2012). Some researchers assert that the Assessment of Teachers’ PCK could be done through micro-teaching (Setyaningrum, Mahmudi, & Murdanu, 2018; Ünver, Özgür, & Güzel, 2020). In the case of pre-service teachers, they have challenges with student’s thinking, mistakes and responding (Korkmaz & Şahin, 2019; Setyaningrum et al., 2018; Ünver et al., 2020). It makes sense as they have limited teaching experiences or even have not taught yet. For in-service teachers, Baumert and Kunter (2013) developed instruments to measure teacher's professional competence (COACTIV). The COACTIV adopted the three main core knowledge CK, PCK and PK from Shulman's work and extended it.

As one of the ways, testing is used to assess teachers. The Ministry of Education and Culture (MoEC) of the Republic of Indonesia has also implemented Teacher Competency Tests (TCT) to evaluate teachers' knowledge. The result of this assessment is both to evaluate teachers and to provide support for them (Widodo & Tamimudin, 2014). However, the content of this assessment is commonly dominated by SMK, in this case within the mathematical problems. It seems that the PCK has not been measured fully through this wide assessment. Another study using testing faced challenges in measuring teachers' knowledge (Fauskanger, 2015). An interesting finding of a study of pre-service teachers is that they possessed higher PCK scores than SMK from limited test items (Kristanto, Panuluh, & Atmajati, 2020). A case study in South Korea revealed that teachers with sufficient SMK of a certain competence/ topic faced challenges in incorporating KCS and KCT of that topic (Lee, Capraro, & Capraro, 2018). Therefore, testing to measure teachers' knowledge still face challenges.

Lesson plans are considered to play an important role in teaching and learning. Having a good lesson plan is important in ensuring that learning would take place during the lesson (Jones & Edwards, 2010). Academics argue that the key determinant of success in teaching is the effectiveness of planning and how well a plan is carried out in the classroom. Effective lesson planning considers possible classroom problems and how to tackle them adequately (Jones & Edwards, 2010). In the common Japanese lesson plan, it contains detailed instruction so that teachers can easily understand it when reading it (Nakahara & Koyama, 2000). Japanese lesson plans also include possible student solutions and errors. The blackboard is also carefully planned. Called, 'Bansho', which anticipates and tries to elicit student mathematical thinking and student thinking schema for solving the given problems.

In developing lesson plans, teachers integrate their knowledge, such as subject matter knowledge and pedagogical content knowledge (An, Kulm, & Wu, 2004; Burns & Lash, 1988; Simon, 1995). A study in Australia revealed that the teacher, in planning a lesson, gave attention to students' engagement (Clarke, Clarke, Roche, & Chan, 2015). The students' engagement involves a choice from many pedagogical strategies, all designed to motivate the students to engage with the topic. It has been shown by several studies that novice teachers improved their PCK by teaching and preparing to teach (Turnuklu & Yesildere, 2007). There is a reciprocal relationship between teacher thought process (including planning) and teachers actions, the latter much influenced by the former (Clark & Peterson, 1986; Superfine, 2008). In other words, teacher classroom practices are influenced by a complex mix of teacher beliefs, attitudes knowledge and intentions Therefore, arguably it is possible to look at teacher lesson plans to investigate their knowledge. The illustration of a model of teacher knowledge and planning can be seen in Figure 2.

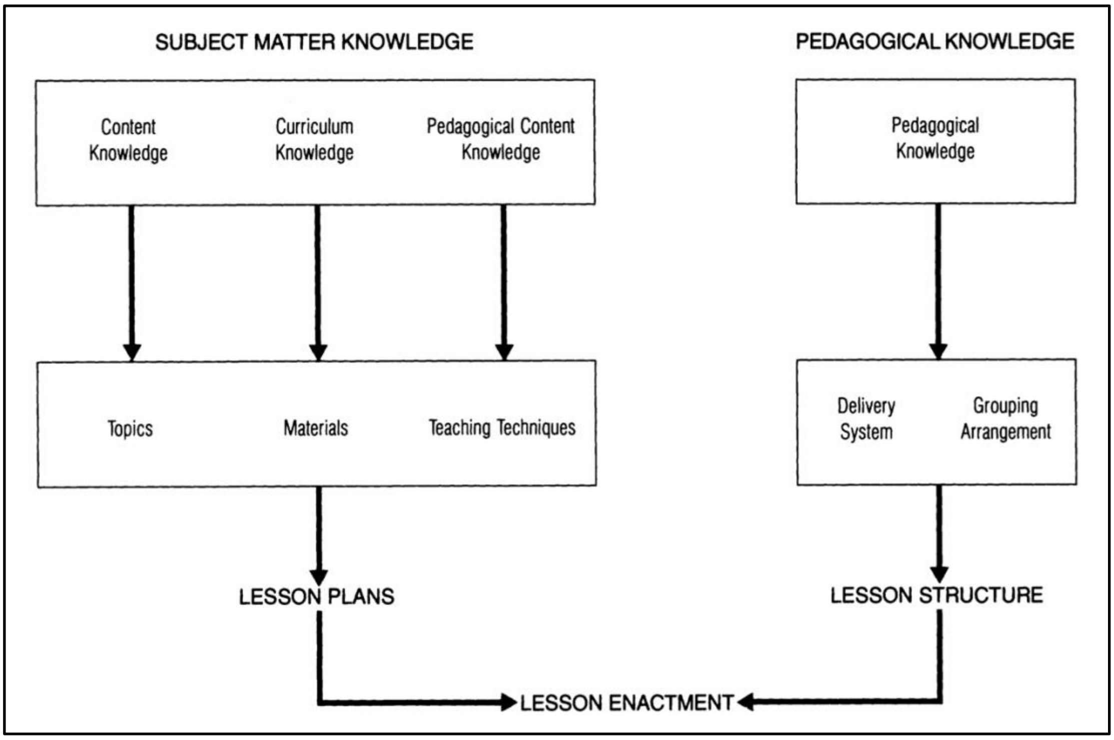

Figure 2. Model of teacher knowledge and planning (Burns & Lash, 1988, p. 382)

Carle (1993) has investigated several student misconceptions related to the area-perimeter topic. A meta-analysis of research has shown some student misconceptions on area measurement was due to area being taught together with perimeter causing many students to confuse area and perimeter (Watson, Jones, & Pratt, 2013; Cavanagh, 2007). Cavanagh (2007) studied Australian Year 7 secondary students and reported students experienced difficulties dealing with area concepts because of the above confusion with perimeter. As a consequence, students used slant and perpendicular height interchangeably. Zacahros & Chassapis, (2012) reported Greek Year 6 elementary students added the base plus the height instead of multiplying base with height to find the area of a rectangle. Özerem (2012) reported that seventh year secondary school students in Cyprus had a number of misconceptions due to a lack of knowledge related to geometry, resulting in them using the wrong formula. This lack of understanding of the concept of area resulted in students memorizing the formulas. Students who learn through manipulating area seem likely to avoid misconceptions on area measurement (Watson et al., 2013). It seems to make sense as they could manipulate and observe what changes happen by reshaping a figure (Yunianto, 2015).

It has been shown that SMK and PCK of mathematics teachers influenced student performance (Baumert et al., 2010). Thus, we should not expect teachers to deliver mathematics well if they do not have mastered it and do not understand how to teach it. Kow and Yeo (2008) explored the importance of SMK and PCK in the topic of area-perimeter from the planning of the lesson to its delivery. It was found that teachers with strong SMK and PCK provided more freedom to students to approach the task. Baturo and Nason (1996) evaluated first-year teacher education student understanding of subject matter knowledge in the domain of area measurement and uncovered many misconceptions. Success was related to their experience of learning the topic. John (2006) argued that novice teachers have difficulty making predictions about student responses and how to respond to unpredicted situations they encountered. In line with this, lack of mathematics pedagogical content knowledge of the teacher potentially lead to students having misconceptions (Kow & Yeo, 2008).

This study intends to focus on a part of PCK, the KCS within lesson plans on the topic of area-perimeter of a rectangle. It is necessary to obtain a fuller insight into teacher knowledge as it influence students' performance. Beside testing, there might be alternative way such as lesson plans to investigate teachers' knowledge. How are mathematics teachers prepare their lesson plans and how is PCK integrated in their lesson plans? How are the KCS integrated in the lesson plans? In the next section, the ways of gaining this insight will be discussed and the strategies used in collecting and analyzing the data. Furthermore, the results and discussion sections will describe the KCS evident in the lesson plans and the interviews with the respondents.

METHOD

This research involved humans and has been approved by IOE research ethics of University College London (IOE.researchethics@ucl.ac.uk) as this is a part of completion of the first author's dissertation. This study administrated a case study approach. This approach suits this study as it does not seek to generalize the findings but to gain deeper insight into the issue (Denscombe, 2010; Yin, 2014). The research subjects were the mathematics teachers in Yogyakarta and its surrounding registered themselves to participate on PD organized by SEAMEO QITEP in Mathematics. Some teachers teach across multi-grades. The first researcher who was facilitating one of the sessions asked the participants to develop a lesson plan as part of the whole PD. It was done somewhere in the middle of all complete sessions. As it is a case study, the researchers examined two selected lesson plans of two mathematics teachers. The remaining lesson plans have not been analyzed due to time limitation. The sample was chosen from twenty-nine teachers who attended a professional development (PD) session, and two teachers were selected for the lesson plan analysis and interview. Additionally. these teachers were selected based on their teaching experience; at least five years. The interview scenario was a semi-structured interview, and the two teachers were interviewed together. The two teachers who had been interviewed were a female teacher and a male teacher. They have different years of teaching experience. The female teacher teaches in a city while the male teacher teachers in a rural area. Participation in this study was voluntarily. The Indonesian mathematics teachers attending this PD were teaching grade 7 to grade 9. The topic that would be taught was area and perimeter for grade 7. The "Gold Rush/Mining" task was selected. This task was chosen because it is a problem-solving task and has several ways to be solved on area-perimeter of a rectangle (see Figure 3). Additionally, the complete Gold Rush activity showed the mistakes that students might do. Thus, it is considered as a good activity to be explored to understand how teachers prepare this activity.

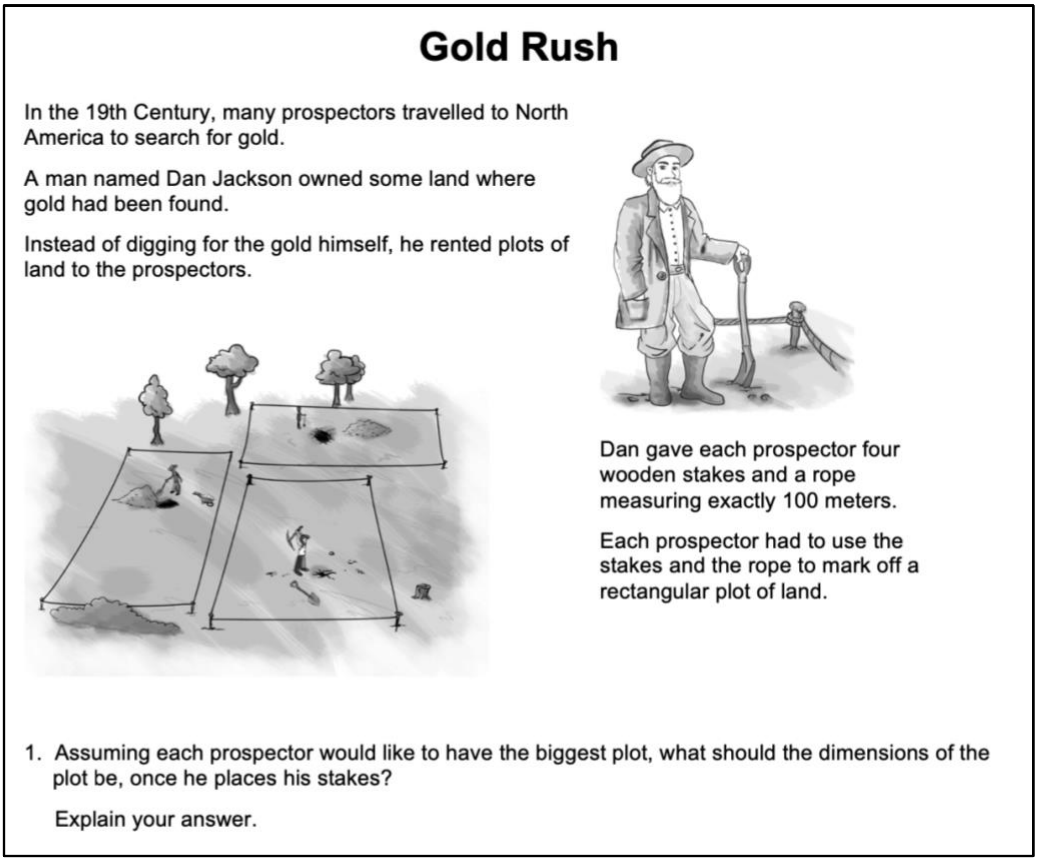

Figure 3. The Gold Rush problem (https://www.map.mathshell.org/download.php?fileid=1637)

To analyze the lesson plans, the researchers used content analysis. This method has the 'potential to disclose many hidden aspects of what is being communicated through the written text' (Denscombe, 2010, p. 282). From the lesson plan, the researcher would investigate to what extent the teachers' knowledge of students' conceptions and misconceptions is reflected in their written lesson plans (Table 1). The two lesson plans were coded to find themes by classifying instructions and KCS integrated in the lesson plans.

Table 1. Knowledge of Content and Student (KCS) (Ball et al., 2008, p. 401)

No.	Knowledge of Content and Student
1.	The ability to anticipate what students are likely to think and what they will find confusing
2.	The ability to predict what students will find interesting and motivating when choosing a task
3.	The ability to anticipate how students are likely to solve a given task and whether they will find it easy or difficult
4.	The ability to hear and interpret students' emerging and incomplete thinking

By using Table 2, it is easy to differentiate instructions' categories. These themes were useful in providing information on what the lesson plans contained. It focused on whether or not, the teachers included information about what students would do to the task (KCS). The data were presented descriptively.

The two lesson plans were coded and analyzed. There were three types of instructions to refer to with the codes. First, general instruction (GI) is where the teacher gives students instructions in a general way. This type of instruction is relatively simple, short and contains the doer(s) and their actions (verb) but leads to some mysteriousness (unclear). The second type of instruction is specific instruction with no detail (SIND). This refers to specific action, which has more information than GI but lacks detail in necessary aspects. The last type of instruction is specific instruction with detail information (SID). This instruction provides more detail and clearer information. Some forms of SID are short and require no detail, as it can be found easily or understood easily in other parts of the text. Looking through the instruction types, the researcher seeks evidence of KCS on the lesson plans (Table 2).

Table 2. Coding for instructions

Code	Example 1	Example 2
GI	Teacher asks a question to students	Teacher asks students to present their work
SIND	Teacher asks a question to students about their strategy.	Teacher asks two groups to present their work
SID	Teacher asks a question to students about their strategy. "what did you do and How did you do it? How are you convinced with your strategies?	Teacher asks two groups with different strategies to present their work starting with the group with less sophisticated strategy.

The two teachers were also interviewed to gain more insight. They were interviewed together (focus-group interview). The researcher wanted to clarify what was written on the lesson plans and why. Through a semi-formal interview style, data were collected through voice recording as well as video recording. From the records, data were transcribed and analyzed.

RESULTS AND DISCUSSION

Using the codes, the lesson plans revealed some interesting findings. Teachers 1 (T1) and Teachers (T2) have different proportions of the use of the instructions (Table 3). The percentage is from type of instruction per total instructions written on the lesson plans.

Indonesian teachers follow the prescribed template of a lesson plan by MoEC. The template consists of three main parts namely; introduction, main and closure. It also consists learning goals and how teachers and students would do in the classroom.

Table 3. Proportions of the instructions

Instruction	T1	T2
GI	8 (35%)	6 (31.6%)
SIND	6 (26%)	7 (36.8%)
SID	9 (39%)	6 (31.6%)
Total	**23 (100%)**	**19 (100%)**

Based on the partition T1 used more instruction in the introduction and has less instruction in the main body. Interestingly, T2 has more instructions in the Main body with detailed information. Compared to T1, T2 had fewer total instructions, and detailed instructions (SID). From T2's SID, there were several instructions that provided information relating to PCK (Table 4).

Table 4. Comparison of Instructions

Code	Introduction		Main		Closure	
	T1	T2	T1	T2	T1	T2
GI	2	0	3	4	3	2
SIND	3	1	3	3	0	3
SID	7	2	1	4	1	0
Total	**12**	**3**	**7**	**11**	**4**	**5**

T1 put more details of what students would ask to her on her lesson plan. For instance: 'Can I solve it freely?' has been put on her lesson plan. This is a proof of PCK in the lesson plan, but not specific to KCS.

- **Main Activity 100 minutes**

  <u>PHASE: Organizing Students</u>

  Students make up groups consisting of 4-5 students.

  - **Observing**

    After receiving the worksheet (problem), students observe the problem within their groups.

  - **Questioning**

    Students ask some questions related to the worksheet such as:

    - I still do not understand what the problem means.
    - Can I solve it freely?

  <u>PHASE: Guiding the individual and group investigation</u>

  - **Gathering Information/ Data/ Trying out**

    Students look for data and discuss the problem on the worksheet of Gold mining.

  - **Reasoning/ Associating**

    Students conclude the result of their discussion.

  <u>PHASE: Developing and Presenting the result</u>

  - **Communicating**

    Students communicate their result in written or oral presentation. One of the members of the group presents the result and other groups respond to him.

- **Closure (10 minutes)**

  <u>PHASE: Analysing and Evaluating the process of problem solving</u>

  1. Teacher facilitates students to conclude what they have just learned)
  2. Teacher facilitates students to identify the parts that they both understand and not.
  3. Teacher gives homework or assignment to students.
  4. Teachers informs students that the next lesson would be about triangle.

Figure 4. Teacher 1 Lesson Plan

In addition, the way she would organize the discussion are provided in detail. This would provide information to other readers/ teachers how the classroom discourse was managed (Figure 4). On the phase of guiding the individual and group investigation which be rich of KCS. In this lesson plan, detail ways of students might solve it or make mistakes and how to facilitate it have not been depicted.

The T2 lesson plan of rectangle using Gold Rush task depicted detailed information about a possible student strategy (KCS). Figure 5 shows that T2 considered one strategy that students would utilize by asking students to make a table. T2 prompted students to make a table and gave an example to start with simple numbers. Within that table students would investigate the largest area by filling the lengths and widths that added to 100. More interestingly, two examples with easy numbers were provided to support students. Therefore, T2's instruction can be understood as providing a method to solve the task, with much support given to students.

Main Activity

- ☑ Teacher divide students into groups
- ☑ Teacher delivers the worksheet to be discussed
- ☑ Teacher facilitates the learning processes

- For the first question, students are asked to make a table by filling up the length column and determine the width to make 100 m.
  for instance, p=10, l=… m
  p = 15 m, l = ...m, then the area = …. p=15 m, l= … m etc
  Students determine the largest area by themselves
- For the second question, after students have solved the largest area for one miner, then how if it is for 2 miners?
  Next, if the ropes of the 2 miners are joined, and continue like the first question, what will be the largest area?
  How if you continue doing this for 3 miners and 4 miners until n miners?

Figure 5. Teacher 2 Lesson Plan of Gold Rush

After finding the largest area of the rectangle, students had to find the largest area by joining two miners' ropes and how would they join it. T2 also offered questions for students, revealing the organization on their lesson plan. T2 has also provided students actions in Figure 6.

- Students evaluate and make generalisation into questioning.
  - ☑ Teacher asks students to present in front of the class
  - ☑ Other students respond the presenter

Figure 6. T2's lesson plan on organizing the classroom discussion

Students were expected to evaluate and generalize during discussion. Although it was unclear what kind of evaluations and generalizations would be made. It would be clear if he put, for instance, that the generalization would be that 'the largest area would always be a square'. This generalization might come out from students. In addition, it was not clear how T2 would organize the presentation, or which group would present first. If there were two groups with different strategies or different conclusions, it is not clear how it would be organized.

Teachers T1 and T2 have more than five-years teaching experience each. Based on the questionnaire and interview, their schools are different in terms of location and students' background. These teachers themselves employed different abilities in solving the Gold Mining problem (Figure 3). From the conversation below, it seems that they have three correct strategies or less to solve it: T1-Ms. Excel integration and T2 -table, quadratic function and graph. However, there is a significant difference between the two teachers. T1 allowed the students to solve the task freely (students' own ways).

The interview with Teacher 1 showed that she has the ability to solve the problem.

R : Are there other ways T1?
T1 : Yesterday, I just did that one.
T1 : …just let students find the ways to solve it …. Then, I will let them know that there are some ways to solve it. I give that opportunity to students

This teacher (T1) would allow her students to approach the task in their own ways. However, T2 had a different way of letting students approach the task, providing only one strategy.

*T2 : To me, I could do it directly because **I already knew it** but to students if I want to students to learn it, **I make a table for them**. If the table is not made, students will find it difficult to solve it for students in my school.*
R : So, you (T2), induce them by using the table?
T2 : Yes, by the table.
R : What do you think, how many ways to solve it?
*T2 : To me, I did one way I know it directly it would be a square**. I knew it already**. But for students, **with table**, students will measure the perimeter, area, so if the length is 5, how long is the width, if the length is 10, how long is the width, and.., they will list it, this is how I let them learn. If I do not do it they will have no clue to solve it.*

From the transcript of T2, he seemed to only allow his students to use one strategy. He believed that his students would not be able to approach the task without inducing the table. He has had previous experiences where students were unable to complete a similar task.

*T2 : I have tried several times an easier task, for instance, given the perimeter of a rectangle and how big is the area, changing from the perimeter to area, I let them do it and facilitated them, but students were not able. For the story problem, the reading comprehension, the task asks to go to the East, most of my students go to the West (**metaphor**).*
T2 : However, I have thought only one strategy, which is global to solve a task. … I, I... know at least I understand my students' characteristic so that it will be difficult for my students. … It is not possible to come up if I let them to do it freely. … I am so careful to give it the various strategies because students would get confuse

To know how to solve the mathematical task, these teachers tried the problem themselves. During the interview, T2 seemed to be familiar with the task and had three ways of finding the answer. Meanwhile, T1 only thought of one strategy.

T2 : By using the strategy of making rectangles with certain sizes and order them and estimate the biggest area.
T2 : To me, I did one way I know it directly it would be a square. I knew it already
T2 : ...instead of table, we can make the variable x, then I will be a quadratic function,
R : Are there other ways to solve it?
T2 : For the time being, not yet, making rectangles and to the square
R : Do you think there are still other ways to solve that problem?
T2 : I could use the graph ...

To some extent, from the lesson plan, T2 gave students a global strategy (table) to solve the task based on his previous experiences, although there is no guarantee that students would continue to have the same issues with the task (Figure 5). However, by giving the students the strategy, he inadvertently is making the students dependent on him. Whereas, from the lesson plan, T1 is helping the students to make decisions themselves (Figure 4). From the interview evidence, the two teachers have different abilities in solving the task and differ on the approaches they offer to their students.

In relation to students' possible mistakes and misconceptions, it seems that these teachers had some ideas as to what their students would find difficult.

T1 : The task has missing information, it should be more, and some students would think that. So that they ***have not thought*** *yet the possible ways to solve it. In average, students can directly solve it with possible ways to do. They can find it directly.*
T1 : 100. Maybe ***they thought that*** *that's the only think they know.*
R : ... So, they would answer it 100, possibly
T1 : Yeah, possibly
T2 : ... for those who did not understand, ***they would not know what 100*** *m rope is to with the perimeter. So that the concept of perimeter, for those who understood, they already make it but later* ***they would not think*** *the rectangles can be varied.*
T2 : Students ***would confuse*** *the meaning of maximum, which is the largest,* ***they have not thought about it.*** *So that students' thinking is not yet there. Their thinking is still circulated on the perimeter not yet the perimeter to area and from area to find maximum area.*

Teachers also have ways of responding to students' mistakes, prompted by the researcher (Figure 7). The researcher proposed a possible mistake by a student of which the shape looks like a rectangle 25 x 26,5.

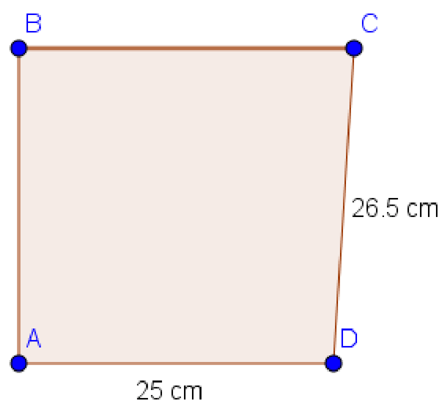

Figure 7. A student's possible mistake proposed by the researcher

If faced with a student mistake that they have not thought of before, both teachers seemed to engage thoughtfully with the scenario presented and sought ways of supporting students in addressing the mistake. Rather than telling a student their answer was incorrect, they asked what the task wants, and told them to check whether the shape is a rectangle or not.

R : If it happens if you see this (showing)
T1 : I would ask students back to try it then you calculate it as what being asked to you
R : They have not yet known the result!
T1 Try, try it, by trialing they would know that it is different, this one is more, and that one is like that, ...
R : T2, what if your students did this? what would you do?
T2 : I would check it first, is it correct or not, the shape is a rectangle or not, they said that it is not, so I asked whether the perimeter is 100 cm or not. So, by knowing that it is a rectangle, the length would be equal, and the width would be equal (opposite sides), so that the perimeter would be 100 cm...

In this study, the lesson plans facilitated an insight into teachers' knowledge. In this case, it showed teacher's pedagogical knowledge as well as PCK. Lesson plans can contain rich information on how the lesson is expected to be carried out. This is potential data to be used for assessing teachers' knowledge. How the teachers organize and manages the classroom, task, and the discussion would be depicted in the lesson plans. This resonates with Burns and Lash (1988) and Simon (1995) who argue that in developing lesson plans, teachers integrate their knowledge, such as SMK and PCK. On the other hand, experienced teachers may not use paper planning (written lesson plan) or just outlines as they have knowledge of what will work best (Butt, 2008; Jones & Edwards, 2010). In addition teachers also do mental planning for the lesson plans and the lesson plans are not written (Borko, Livingston, & Shavelson, 1990). The dynamics of a classroom are very fluid, and a teacher must adjust to that fluidity while following the plan. It is rare for a lesson to go exactly to plan. Yet, the execution of the lesson plan determines the effectiveness of the lesson (Kow & Yeo, 2008). In Japanese lesson plans, they contain more detailed instructions (Nakahara & Koyama, 2000) which shows more information about teachers knowledge. In contrast, the two case of teachers in this study, have not yet shown detailed instructions but more in general instruction.

Teachers have different ways of supporting students to solve tasks (Yeo, 2008). Students' performance is more affected from teachers' PCK (Baumert et al., 2010). However, SMK is basis knowledge for teachers (Shulman, 1986; Turnuklu & Yesildere, 2007). It is not usual that teachers teach 'something' before mastering the subject matter thus reducing the possibility of teaching effectively (Turnuklu & Yesildere, 2007). The teachers in this study were able to solve the task and had some ways to respond to students when they made mistakes in solving the given task (possessing SMK and PCK). However, these results are not generalizable. The limited sample was not chosen randomly and as these teachers came from relatively developed areas in Java and have at least five years teaching experiences they are not representatives of the wider Indonesian teaching population. Mathematics teachers in this study might not show detail information on their lesson plans and have not fully been aware of integrating PCK on developing their lesson plans. This study might not cover all mathematics teachers' PCK profile in Yogyakarta or broadly in Indonesia. However, this study has provided an interesting glimpse into one part of the very complex decision and knowledge processes that are involved in teacher pedagogical knowledge.

CONCLUSION

This study indicates that it is possible to assess teachers' KCS of a specific topic through analysis of the lesson plans when supported by interviews. There is evidence that these teachers had some knowledge about student strategies and misconceptions about the area-perimeter of rectangle topic, and that this knowledge was not necessarily fully integrated into their lesson plans. When prompted to think about possible misconception, the teachers found that it was challenging. Understanding possible misconceptions, making predictions and the anticipation of student responses would help teachers to be better prepared in facing the situations during teaching. Developing problem solving skills and autonomy among students requires teachers to stop providing a particular way (limiting students' strategies) but rather provide an environment where students are able to choose strategies, to make mistakes and to explore. Training for teachers could be more supportive in providing pedagogy that promotes such an environment. Additionally, this study explored a rectangle topic, the result might vary in different topics. Therefore, further investigation on different topic could be conducted. This study is not generalizable as it used limited research subjects.

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SECONDARY SCHOOL MATHEMATICS TEACHERS' PERCEPTIONS ABOUT INDUCTIVE REASONING AND THEIR INTERPRETATION IN TEACHING

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Abstract

Inductive reasoning is an essential tool for teaching mathematics to generate knowledge, solve problems, and make generalizations. However, little research has been done on inductive reasoning as it applies to teaching mathematical concepts in secondary school. Therefore, the study explores secondary school teachers' perceptions of inductive reasoning and interprets this mathematical reasoning type in teaching the quadratic equation. The data were collected from a questionnaire administered to 22 teachers and an interview conducted to expand their answers. Through the thematic analysis method, it was found that more than half the teachers perceived inductive reasoning as a process for moving from the particular to the general and as a way to acquire mathematical knowledge through questioning. Because teachers have little clarity about inductive phases and processes, they expressed confusion about teaching the quadratic equation inductively. Results indicate that secondary school teachers need professional learning experiences geared towards using inductive reasoning processes and tasks to form concepts and generalizations in mathematics.

Keywords: Perception, Inductive Reasoning, In-Service Mathematics Teachers, Secondary School

Abstrak

Penalaran induksi merupakan hal yang penting di pembelajaran matematika untuk membangun pengetahuan, pemecahan masalah, dan membuat generalisasi. Namun, baru sedikit penelitian yang telah dilakukan tentang penalaran induksi yang diterapkan di pembelajaran konsep matematika di sekolah menengah. Oleh karena itu, studi ini mengeksplorasi persepsi guru di jenjang sekolah menengah tentang penalaran induksi dan menjelaskan tipe penalaran matematika di pembelajaran persamaan kuadrat. Data dikumpulkan dari kuesioner terhadap 22 guru dan dari interview untuk mendapatkan jawaban lebih dalam. Melalui metode tematik analisis, ditemukan bahwa lebih dari separoh guru ini memahami bahwa penalaran induksi adalah suatu proses dari hal khusus ke umum dan sebagai cara untuk mendapatkan pengetahuan matematika melalui bertanya. Dikarenakan guru ini memiliki kejelasan tentang fase penalaran induksi dan prosesnya, mereka mengalami kebingungan tentang mengajarkan persamaa kuadrat secara induksi. Hasil penelitian ini menunjukkan bahwa guru di jenjang sekolah menengah ini membutuhkan pengalaman pembelajaran professional tentang penggunaan proses penalaran induksi dan penugasan untuk membangun konsep dan generalisasi di matematika.

Kata kunci: Persepsi, Penalaran Induktif, Guru Matematika, Sekolah Menengah

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The processes of the knowledge discovery and construction of proofs in mathematics involve both inductive and deductive reasoning (Davydov, 1990; Lee, 2016). The first implies moving from the particular to the general, and the second moves from the general to the particular (Hodnik & Manfreda, 2015). This work focuses on inductive reasoning, although some teachers are accustomed to employing deductive reasoning to teach mathematics (Rott, 2021). Siswono, Hartono, and Kohar (2020) defined deductive reasoning as 'a process of deducing conclusions from known information (premise) based on formal logic rules, where the conclusions must come from information provided and do not need to validate them with experiments' (p. 419). Inductive reasoning, on the other hand, is oriented to infer laws or general conclusions through observation and connection of particular instances (be they facts, premises, or particular cases of situations or a class of mathematical objects), and the conclusions can be verified by experimentation (Haverty, Koedinger, Klahr, & Alibali, 2000; Polya, 1957). According to Reid and Knipping (2010), three invariant characteristics of this type of reasoning are that it (a) comes from specific cases to conclude general rules, (b) uses what is known to conclude something unknown, and (c) is only probable but not certain.

Inductive reasoning has a core function in intellectual processes development for mathematics (Klauer & Phye, 2008; Mousa, 2017; Tomic, 1995). This type of reasoning is particularly important for learning mathematics in primary and secondary school, for two reasons. Firstly, it constitutes a teaching pathway for developing concepts and solving mathematics problems (e.g., Molnár, Greiff, & Csapó, 2013; Christou & Papageorgiou, 2007; Sriraman & Adrian, 2004). Inductive reasoning contributes to the formation of concepts because it 'lead[s] to detecting regularities, be it classes of objects represented by generic concepts, be it common structures among different objects, or be it schemata enabling the learners to identify the same basic idea within various contexts' (Klauer, 1996, p. 53). Secondly, it is one of the forms of reasoning that supports the process of generalizing numerical and figural patterns or mathematical objects (Cañadas, Castro, & Castro, 2008; 2009; Rivera & Becker, 2016).

The National Council of Teachers of Mathematics [NCTM] (2014) established that this form of mathematical reasoning must progress in students throughout each education level so that they can become more proficient in formulating conjectures and generalizations from specific cases. For that reason, secondary school teachers should develop and interpret the students' reasoning (AMTE, 2017; NCTM, 2020). However, several studies have reported that pre-service and in-service teachers have difficulties in solving generalization tasks from particular cases through inductive reasoning (Rivera & Becker, 2003; 2007; Sosa & Aparicio, 2020). In particular, they show difficulties associated with establishing a pattern and achieving the abstraction of the general when solving quadratic pattern tasks (Manfreda, Slapar, & Hodnik, 2012; Sosa, Aparicio, & Cabañas, 2019; 2020).

In this sense, knowing the type of perceptions that teachers have about inductive reasoning and how they promote it in teaching is essential to address these difficulties. Some studies suggest that promoting and interpreting inductive reasoning in the classroom is a complex task for teachers. Herbert, Vale, Bragg, Loong, and Widjaja (2015) reported that elementary school teachers have little understanding of the distinctive aspects of the mathematical reasoning types and how to encourage mathematical reasoning in the classroom. Furthermore, noticing and interpreting the actions of students' reasoning in generalization tasks is complicated for both pre-service and in-service teachers (Callejo & Zapatera, 2017; El Mouhayar, 2018; Melhuish, Thanheiser, & Guyot, 2018). De Koning, Hamers, Sijtsma, and Vermeer (2002) claimed that elementary school teachers have difficulty focusing on the inductive process when teaching mathematical structures because attention is paid to the content or to students' responses and not to the process itself.

Rott and Leuders (2016) reported that teachers have the epistemological belief that inductive reasoning justifies discovery in mathematics over deductive reasoning at a 2:1 ratio. Recently, Rott (2021) showed that an inductive belief prevails over a deductive belief in more than half a group of secondary school teachers, but those teachers did not provide arguments for their belief. According to this author, it is necessary to investigate the consequences of such epistemological beliefs in the mathematics teaching. To provide information in this direction, the aim in our study was to analyse secondary school teachers' perceptions about inductive reasoning associated with their interpretations of this reasoning in teaching the quadratic equation.

Negative, or inadequate, perceptions of teachers concerning mathematics could unfavourably affect students' learning (Rosli et al., 2020). Therefore, our study contributes to identify whether the teachers' perceptions about inductive reasoning are adequate or not to encourage this type of reasoning in their students. It is desirable that teachers have clarity about inductive reasoning phases that go along with the transition from the particular to the general for discovering properties, knowledge, and general rules in mathematics.

In this regard, some authors have pointed out the phases and inductive processes people use to generalize from particular cases. Polya (1967) proposed four phases: observation of particular cases, conjecture formulation, generalization, and conjecture verification. Cañadas and Castro (2007) developed an empirical model of secondary school students' inductive reasoning that expands the phases referred to by Polya and comprises the following seven phases: working with particular cases, organisation of particular cases, search for and prediction of patterns, conjecture formulation, generalization, and demonstration. Sosa, Aparicio and Cabañas (2019) reported that mathematics teachers managed to generalize inductively when they connected three cognitive processes: observation of regularities, the establishment of a pattern, and generalization formulation.

We assume that if teachers have inadequate perceptions or little understanding of inductive reasoning, they will have difficulty in promoting this reasoning in teaching. Besides, there is a gap in the literature concerning the secondary school teachers' perceptions about inductive reasoning, even when this type of reasoning is a means of mathematical learning and it is possible to develop it starting in elementary school (Molnár, 2011; Molnár, Greiff, & Csapó, 2013; Papageorgiou, 2009). These factors led us to ask: What are secondary school teachers' perceptions of inductive reasoning? And how do they interpret it in teaching the quadratic equation concept?

METHOD

This research is qualitative, exploratory, and interpretative. It is exploratory because perception and teaching of inductive reasoning of in-service teachers is a little-studied topic. There are only a few approaches to this topic from a cognitive perspective or from teachers' epistemological beliefs in the literature. An interpretative approach was considered to generate categories of teachers' perceptions and to identify ways in which teaching is carried out by interpreting and making sense of the characteristics attributed to inductive reasoning by teachers in a written and an oral way (Freitas, Lerman, & Park, 2017). The collection of data on the perception and interpretation of the teachers was carried out with an open questionnaire and an interview, both written and oral.

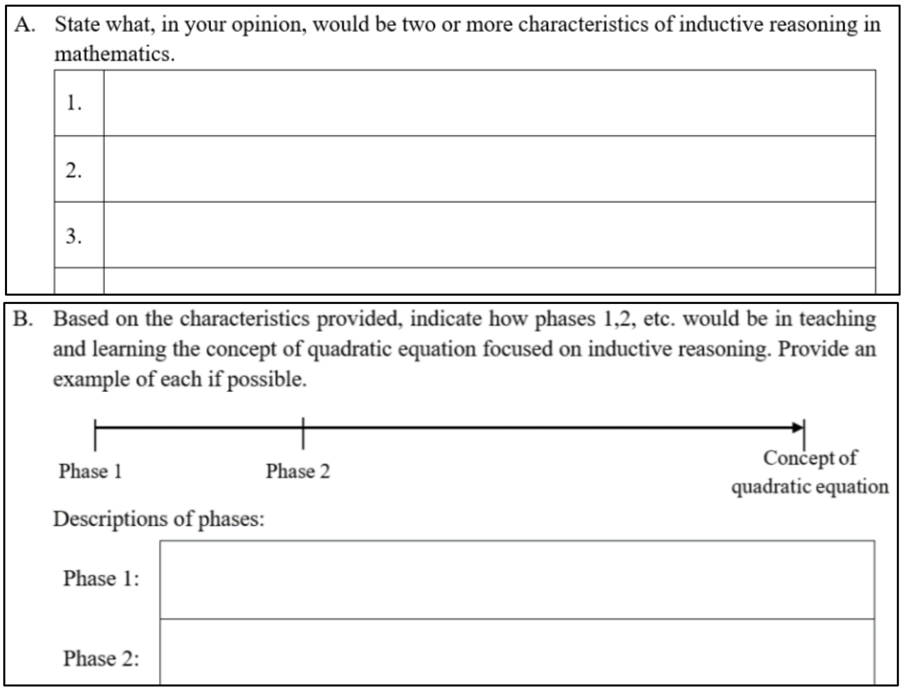
Context and Participants

This study was conducted with the participation of secondary school in-service mathematics teachers from Mexico; they were invited to participate in a professional teacher development program in mathematics through an open call. The program aimed to develop the teachers' inductive reasoning and to encourage them to enact this kind of reasoning in learning activities. Before the program began, 22 teachers—14 women and eight men—were selected among the teachers enrolled in the program; they agreed to participate in the study. The criteria for their selection were: (i) to have at least one year of experience teaching patterns and quadratic equations; (ii) to have the mathematical knowledge to solve tasks of generalization of quadratic patterns by inductive reasoning, whether acquired during their professional training or in training courses for teachers; and (iii) to know about inductive reasoning and mathematical generalization. The data for the selection of the participants were obtained from the academic information given by the teachers on the registration sheet for the program.

These criteria are explained by the fact that mathematics teachers have difficulties in generalizing quadratic patterns, as is reported in the literature. Besides, inductive reasoning is one of the mathematical reasoning types necessary to solve quadratic pattern generalizing tasks (Cañadas, Castro, & Castro, 2009; Rivera & Becker, 2016). The quadratic equation concept was chosen because, in the mathematics curriculum in Mexico, it is associated with the activity of generalizing quadratic patterns (Ministry of Public Education, 2017). In relation to the mathematical standards of the NCTM (2014), the aim of the mathematical activity in secondary school in Mexico (grades 7–9, ages 12–14) is to develop abilities such as generalization; abstraction; and inductive, deductive, and analogical reasoning. The students are expected to learn how to model linear, quadratic situations and to define patterns through algebraic expressions (Ministry of Public Education, 2017).

Data Collection

The data were collected in two working sessions. In the first one, the teachers gathered in a classroom and were asked to answer a written, open questionnaire, individually and simultaneously. This questionnaire was used to collect the responses of the participants about their perceptions of inductive reasoning and how these perceptions were brought to teaching. The questionnaire had two items, A and B, as shown in Figure 1. To obtain information about inductive reasoning perceptions, item A asked the teachers to write at least two characteristics of the reasoning in mathematics. Item B was oriented toward increasing understanding of how teachers interpret inductive reasoning to teach a mathematical concept. Therefore, item B asked participants to describe the phases to be followed to teach some aspect of the quadratic equation concept in an inductive way. The responses given by the teachers to this item were expected to be based on an interpretation of the characteristics mentioned in item A.


A. State what, in your opinion, would be two or more characteristics of inductive reasoning in mathematics.

1.

2.

3.

B. Based on the characteristics provided, indicate how phases 1,2, etc. would be in teaching and learning the concept of quadratic equation focused on inductive reasoning. Provide an example of each if possible.

Descriptions of phases:

Phase 1:

Phase 2:

Figure 1. Questionnaire for data collection

Unlike closed or multiple-choice questionnaires, which contained predetermined responses given by the researcher that can skew the thinking of the study subjects, an open questionnaire contributes to a broader and more genuine picture of the perception of the participants (Ashton & Roberts, 2006; Peterson, 2000; Zohrabi, 2013). Thus, the teachers were asked to answer the written questionnaire to give them a greater opportunity to express themselves freely and to correct or complete their answers. There was no time limit for answering the questionnaire. This questionnaire was an adaptation of a previous questionnaire administered to a group of teachers with characteristics similar to those of the participants in this study to explore whether they knew the content (inductive reasoning and quadratic equation) of items A and B, and whether they understood what was requested.

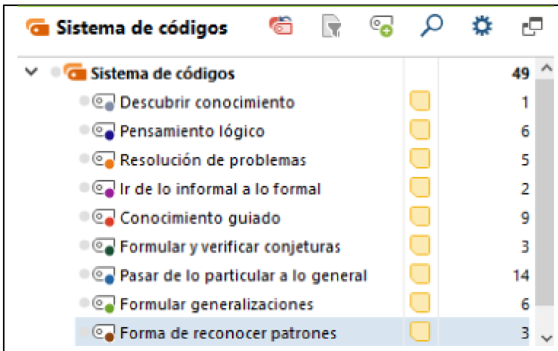
The participants were called for an interview in the second session. During the interview, one of the researchers (first author of this paper) posed questions in an individual and ordered manner to the participants about some words, phrases, or sentences that the teachers used in their responses to the questionnaire. The purpose of this interview was to expand, clarify, or verify their written information and to avoid ambiguities or inadequate interpretations of the written responses on the part of the researchers. The audio of the answers during these interviews was recorded and transcribed for the researchers' analysis, together with the data obtained from the questionnaire.

Data Analysis

A thematic analysis was conducted to describe the teachers' perceptions based on the written and oral answers to item A. The result of this analysis was the generation of categories of teachers' perceptions about inductive reasoning. Then, the responses given to item B were associated with these categories and contrasted with the conceptual framework to identify how teachers interpret inductive reasoning in teaching the quadratic equation concept.

The thematic analysis method consists of identifying, analysing, organising, and systematically obtaining patterns (themes) in a data set by detecting and making sense of the experiences and meanings shared in a group (Braun & Clarke, 2006; 2012). This method was used to identify patterns of meanings in the common characteristics that teachers attribute to inductive reasoning and to form categories related to their perceptions. The six phases of the method were as follows: 1) familiarising yourself with the data, 2) generating initial codes, 3) searching for themes, 4) reviewing potential themes, 5) defining and naming themes, and 6) producing the report (Braun & Clarke, 2012, pp. 60–69).

Phase 1 of the analysis consisted of repeatedly reading the written answers to item A and repeatedly listening to the audio with oral responses. This phase helped in developing an initial overview and making notes about the teachers' ideas concerning inductive reasoning. In phase 2, codes were assigned to extracts of written responses and audio transcripts with key phrases or with characteristics of the inductive reasoning mentioned by the teachers; nine codes were obtained (Figure 2). It should be clarified that a teacher's response could refer to different perceptions of reasoning. The response included more than one code in these cases, and therefore, the number of codified excerpts was larger than the number of participants. In phases 2, 3, and 4, MAXQDA (2018.2) software was used to encode data, group the excerpts of responses by codes to look for themes, and search for the ones with potential.



Code system:

- Discover knowledge
- Logical thinking
- Solving problems
- Go from informal to formal
- Guided knowledge
- Conjecture formulation
- Go from particular to general
- Formulate generalizations
- Way to recognise patterns

Figure 2. List of codes generated in MAXQDA (2018.2)

During phase 3, the generated codes and the excerpts associated with them were grouped and reviewed to search for themes that represent possible categories of the perception of the inductive reasoning of the teachers. For example, the codes 'way to recognise patterns,' 'formulate generalizations,' and 'formulate and verify conjectures' were grouped to form a category that refers to generalizations' formulation and verification.

In phase 4, the themes were recursively reviewed in the context of the codes and total set of responses. The members of the research team became involved in the review and exchange of information during the codification process, searching for themes and defining categories. The main author of this work carried out the first part of these processes in each phase. Another researcher reviewed the generated information later, and finally, the team came together to define the codes, themes, and final categories. In this way, during phase 5, the five categories concerning the teachers' perceptions about inductive reasoning were defined and named. Categories were defined by selecting excerpts of the responses to analyse, clarify, and exemplify each category and to name the resulting categories. Finally, a report for this paper (phase 6) was generated.

RESULTS AND DISCUSSION

The thematic analysis resulted in the detection of five categories of secondary school teachers' perceptions about inductive reasoning. These categories represent patterns of shared meanings among the participants, according to the characteristics that they attributed to this type of reasoning. The following sections present the title, a brief description, and some excerpts of representative responses for each participant perception.

Categories of Perception about Inductive Reasoning

Category A: Way to Acquire Mathematical Knowledge

Teachers perceive inductive reasoning as a pedagogical method of leading students to achieve new knowledge. For them, inductive reasoning consists of posing a problem and, based on the students' previous knowledge, asking key questions so that students acquire new knowledge, similar to the Guided Discovery learning (Honomichl & Chen, 2012). The following excerpts are examples of this perception:

> Teacher C: *It involves the use of previous knowledge so that it can be applied in a more complex situation or to generate new knowledge.*
>
> Teacher L: *Give students an exercise and, based on their previous knowledge, allow them to draw their own knowledge. Have students brainstorm to learn what they know.*
>
> Teacher M: *One of the characteristics is to begin asking key questions for the exercises and introducing students to the topic. Students begin to reason about the topic through questions and can visualise the previous knowledge. Guide questions. During the class, doubts may emerge (...) and questions may be asked to reinforce the student's reasoning (...), students can achieve the appropriation of concepts, processes, etc.*
>
> Teacher V: *Students can come to a conclusion or definition based on their ideas or previous knowledge.*

These were the teachers' predominant perception, though they differ from the function of inductive reasoning as a teaching pathway for the formation of mathematical concepts. The difference is that the teachers did not perceive the function of reasoning as recognising the particular characteristics or attributes of the concept from a set of situations and encapsulating it in a general attribute (Davydov, 1990; Klauer, 1996; Sosa, Cabañas, & Aparicio, 2019); instead, they described issues of the guided discovery so the students could organise and generate their own knowledge through interrogation and group discussion (Yurniwati & Hanum, 2017). This category shows that teachers do not perceive the relationship between the underlying cognitive processes of inductive reasoning and mathematical procedures as something central to the acquisition of new knowledge.

Category B: Cognitive Process

In this category, the teachers perceived reasoning as a process that allows moving from particular instances (e.g. ideas, particular cases, or specific situations) to infer a general conclusion or result. More than half the teachers revealed an adequate perception of inductive reasoning as a cognitive process that involves inferring laws or general rules through observation of particular instances (Haverty et al., 2000). The following excerpts show this perception:

Teacher B: *Start from particular cases to get to general cases. Other cases that meet the observed characteristics are obtained. Conjectures about the observed cases are formulated.*
Teacher E: *It goes from the particular to the general.*
Teacher N: *It is a type of reasoning that consists of moving from particular to general ideas. Starting from concrete ideas to ideas in general. Generalize based on experiences of the given results.*

This perception, very common among teachers, concerns an inherent characteristic of inductive reasoning: It goes from the particular to the general (Reid & Knipping, 2010). Teachers perceive the starting and ending point of inductive reasoning; generalization is recognised as an intrinsic element for this type of reasoning. However, they little or nothing allude to the specificity of the processes that allow continuous progress from the particular to the general; only a few participants described inductive phases or processes such as observing regularities, establishing patterns, and the formulation of generalizations (Polya, 1967; Sosa, Aparicio, & Cabañas, 2019).

Category C: Generalizations Formulation and Verification

Almost a quarter of the teachers associated inductive reasoning with the formulation of generalizations and referred to the experimental character of this reasoning to verify the produced generalizations (Polya, 1967; Soler-Álvarez & Manrique, 2014). That is, they perceive that inductive reasoning is associated with the ability to predict the overall behaviour of specific cases and verify their truthfulness.

Teacher responses referred to how to obtain a generalization and verify it, as can be seen in the following excerpts:

Teacher A: *[In inductive reasoning] students analyse certain characteristics that are repeated continually under specific conditions. It is that they achieve generalizations, establish some rule or generalization based on what is repetitive, verify the established statements (…) I believe that you can establish a statement, and it could be wrong, after the verification; if you said that it was continually happening, for example, for positive numbers, and something else happens for negative numbers(…), you must to prove that it is always repeating; but if you find a case that does not go in the same way, then the generalization will not work. It is like testing if this is real, if this is true.*

Teacher B: *After seeing specific and concrete cases, you can try to predict what is coming next—for example, in a sequence, make conjectures and try to prove them. Predict those conjectures, see if they can be proved, and finally, come to a generalization.*

Bills and Rowland (1999) argued that inductive reasoning is a means for producing mathematical generalizations from particular cases. Thus, Category C differs from Category B, in the sense that reasoning is characterised in terms of generalization as a product of the process of inductive reasoning (Klauer, 1990). According to Fernández-León, Gavilán-Izquierdo, and Toscano (2021), in-service teachers are used to informal reasoning (based on examples) in the justification and generalization (or conjecture) processes. This could be the case because the teachers have a slightly superficial perception of this reasoning type as a means for mathematical generalization. Therefore, inductive reasoning is only perceived globally as a means for predicting and proof in mathematics, but the punctual aspects of this reasoning are not considered. Broadening this perception could help teachers with the development and identification of mathematical conjectures based on empirical data (Cañadas, Deulofeu, Figueiras, Reid, & Yevdokimov, 2007).

Category D: Strategy for Solving Problems

In this category, teachers associate inductive reasoning with problem solving and perceive it as a strategy for obtaining and arguing for the solution. Some of the excerpts where this perception was identified follow.

Teacher P: *They are the premises that allow us to conclude the resolution of problems. It is the form of reasoning that allows us to argue the resolution of problems by induction.*

Teacher H: *Each student should try to solve the posed problem with his previous knowledge.*

Teacher O: *Considering hypotheses or some propositions as starting points to solve a problem.*

Teacher Q: *Establishing a process of resolution based on several cases or examples.*

Inductive reasoning is a useful strategy for solving mathematical problems of categorization, number series, similitude between objects and relationships, and generalization, among other problems (e.g., Csapó, 1997; Molnár, 2011; Tomic, 1995). Furthermore, it facilitates recognition of similitudes in the structure of mathematical problems and generalization of methods of resolution when students work with situations that have different contexts but the same underlying structure (Sriraman & Adrian, 2004). Nevertheless, as in the study of Herbert et al. (2015), the teachers seem to be less aware of the relationship between mathematical reasoning—inductive, in our case—and problem resolution than they were in the previous categories. In particular, we find that these teachers omitted the description of the inductive strategy for solving a mathematical problem; neither pointed out the potential of this reasoning for recognising methods of solving problems that have the same structure. As a consequence, this limited perception might not be enough to promote problem solving skills in students.

Category E: Logical Thinking

Some teachers perceived inductive reasoning as a part of logical thinking—that is to say, as a way of reasoning based on rules and the performance of orderly and coherent procedures. They mentioned the following relevant characteristics:

Teacher J: *It emerges as part of a logical thinking process.*
Teacher R: *Reasoning must be logical—I mean, in an orderly and coherent way. It must follow certain rules to carry out the exercises.*
Teacher S: *It [inductive reasoning] is that the students develop logical thinking, that they understand what they do and perform the procedures in order.*

Category E suggests that the teachers must have a broader perception of inductive reasoning in logic such that they identify and establish inferences based on particular premises and recognise the probable character of the obtained conclusions or propositions (Hayes, Heit, & Swendsen, 2010; Reid & Knipping, 2010). This perception is associated with the fact that the teachers envision inductive reasoning as insufficient for validating mathematical propositions and believe that deductive proofs are needed (e.g. Conner, Singletary, Smith, Wagner, & Francisco, 2014; Martinez & Pedemonte, 2014).

The five categories of perception of inductive reasoning reveal that it is perceived in a very general way as a means and as an instrument to guide the acquisition of new knowledge and to make generalizations. However, the importance of the elements that constitute this form of mathematical reasoning is overlooked, specifically, the observation of regularities, the recognition of patterns, and the formulation of a generalization. Thus, while most of the teachers perceive generalization as a process and product of inductive reasoning, very few show clarities in this sense.

Interpretation of Inductive Reasoning in Teaching

The teachers' description of the phases for teaching quadratic equations led to the identification of four different interpretations of inductive reasoning in teaching. Two of these interpretations are associated with the perception of inductive reasoning as a way to acquire knowledge (Category A) and as a cognitive process to move from the particular to the general (Category B). The other two observed that the ways of teaching in the responses of the teachers are not inductive in nature; one of these forms belongs to deductive reasoning, and the other one was named iconic. Table 1 shows each interpretation and the number of teachers that expressed each interpretation.

Table 1. Inductive Reasoning Interpretations in the Teaching of Quadratic Equations

Interpretation	Frequency	Teachers
Way to acquire knowledge	8	C, E, F, K, L, M, S, T
Inductive (from the particular to the general)	4	B, I, N, R
Deductive (from the general to the particular)	6	A, D, O, P, U, V
Iconic	4	G, H, J, Q

Eight teachers' interpretation was that teaching a concept focused on inductive reasoning consists of guiding students to move from an existing or informal knowledge to a new knowledge, mainly through questioning or examples. This was the case for teacher M explained in Table 2.

Table 2. Phases for Teaching Quadratic Equation Proposed by Teacher M

Phase	Description
1	*Previous knowledge: Introductory questions about algebraic expression, algebraic language, power, the law of exponents.*
2	*Application of the concept of 'basic' shape areas (with square shapes).*
3	*Delete data and replace it with literals. Start with formulas.*

The phases proposed by teacher M are coherent with her perception of inductive reasoning as a way to acquire knowledge. She verbally emphasised the importance of starting with the previous knowledge of the students and using questions to guide them to the definition and expression of a quadratic equation:

> *Recover their previous knowledge; tell them that they had already worked with linear equations, but that there are other types of equations. After, I propose a daily life situation that leads students to represent a square; then I'm going to ask questions that guide them to the relationship of the figure with the formula of the area and make them pose the equation. Tell them that it is the quadratic equation.*

Certainly, inductive reasoning is a way to generalize knowledge by making inferences about unknown cases and new situations based on existing knowledge (Hayes, Heit, & Swendsen, 2010), but the way teachers consider incorporating it into teaching is not appropriate. Even when the teachers interpreted inductive reasoning as a way of teaching to acquire new knowledge, the phases proposed for teaching quadratic equations did not include inductive actions designed to recognise the structure of the equation in different situations or contexts that could allow for the identification of an essential quality of the concept (Davydov, 1990; Klauer, 1996), such as the quadratic behaviour of the variables.

A minority of participants (only four teachers) described the teaching phases in line with inductive reasoning. These phases involve actions concerning the observation of particular situations; the search for and recognition of invariant characteristics of the situations; and a generalization based on a formula, equation, or definition (Cañadas & Castro, 2007; Polya, 1967). For example, teacher B proposed four phases (Table 3) associated with the phases mentioned by Polya (1967). The phases indicated a way to move from the particular to the general, even when the teacher did not specifically refer to the quadratic equation or give examples to illustrate the phases.

Table 3. Teaching Phases based on Inductive Reasoning Described by Teacher B

Phase	Description
1	*Specific cases or situations that can be quantified, manipulated, or visualised are provided.*
2	*Different cases that meet the observed characteristic or property are proposed.*
3	*It is required to predict that this characteristic or property will be fulfilled for other cases that are not tangible or directly observable.*
4	*A rule or formula that covers all possible cases is obtained—that is, a generalization.*

Although the teachers perceived the transition from the particular to the general as a feature of this reasoning, the responses reveal a lack of clarity about the underlying processes. In this way, we consider that these teachers, like the elementary school teachers (De Koning et al., 2002), need instruction about questions or tasks that allow them to shift from the content per se and focus on the processes to develop inductive reasoning in the classroom.

On the other hand, six teachers evidenced confusion about teaching based on inductive reasoning, since the order of the proposed phases refers to deductive reasoning (from the general to the particular). That is, the first phase presents a definition, characteristic, or general formula of the quadratic equation, and the other phases lead to something particular, be it an example or the solution of a specific quadratic equation. Table 4 shows the responses of teacher V to illustrate this deductive interpretation in teaching quadratic equations.

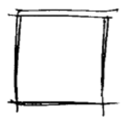
Table 4. Teaching Phases according to Deductive Reasoning Described by Two Teachers

Phase	Teacher V
1	*The characteristics of the quadratic equations are shown to the student.*
2	*Some examples are then shown; students will have to associate them with the quadratic equation of the form* $ax^2 + bx + c = 0$
3	*The student will have to find the unknown in the equation through factorisation.*
4	*a) First, the equation is written* $ax^2 + bx + c = 0$. *Example:* $3x^2 + 2x + 8 = 0$
	b) The expression is then factorised in linear factors. $(3x - 4)(x + 2)$
	c) Set each factor to zero: $3x - 4 = 0, x + 2 = 0$
	d) *Find the value of* x: $x = \frac{4}{3}, x = -2$

Although teachers have a theoretical knowledge of inductive reasoning, it is insufficient to enable them to carry it out in their teaching practice; they are more familiar with teaching using a deductive approach than an inductive one. Even when pre-service and in-service teachers tend to believe that the discovery of mathematics knowledge is inductive (Rott, 2021), the common teaching sequences of some teachers are still in line with deductive reasoning.

The teaching phases proposed by four teachers did not differentiate between inductive and deductive reasoning; instead, they offered an iconic type of treatment. Table 5 shows the phases proposed by teacher G as an example of this type of teaching.

Table 5. Teaching Phases for Quadratic Equations Proposed by Teacher G

Phase	Description
1	*Starting with the area of a square, the student must use his previous knowledge. Area (A) equals side by side (l); area equals the square of the side.* $A = l \cdot l$ $A = l^2$
2	*For example, given the figure of a square, what is the length of the side of the square if its area equals 400 square metres? And if the area is 100 square metres?* 
3	*Make the figure bigger, adding different measurements to the sides of the squares so that they form rectangles or a bigger square.*
4	*Then write x instead of the measurement of the side of the square, so its area equals* x^2, *a squared number.*

In these cases, the phases involved the representation of a quadratic equation by a square or rectangular figure and, sometimes, a squared number. The teachers could have used this geometrical approach and inductive reasoning to obtain a general property: Every quadratic equation may be expressed as the product of linear factors of its roots; but the teachers focused on associating the degree of the equation with the area of squares and rectangles.

The types of interpretation of inductive reasoning in the teaching of the quadratic equation are consistent with the categories of perception. Although the secondary school teachers perceived positive qualities of inductive reasoning in teaching and in the mathematical thinking of the students, most of the teachers in the group showed confusion or an inadequate interpretation of this reasoning when describing the teaching phases. Consequently, most of the teachers' interpretations are inadequate to foment the acquisition of the quadratic equation concept through this reasoning as they relegate the associated phases.

This could be the case because the teachers ignore the principles that guide the development of mathematical reasoning in the students based on generalizations and justifications in classroom (Mata-Pereira & da Ponte, 2017). In addition, the data suggest that the teachers are not aware of the processes (search for attributes and relationships, comparison of similitudes and differences among attributes, resolution and control) involved in the connection of knowledge in an inductive way (De Koning & Hamers, 1999; De Koning et al., 2002; Klauer, 1996).

CONCLUSION

This research identified five perceptions about inductive reasoning among secondary school teachers, which reflects that little clarity and sensibility are present regarding this type of reasoning in the teaching of mathematics. Although these perceptions are positive, teachers need to enhance their interpretations of inductive reasoning if they are to develop such reasoning in the classroom. Results suggest that it is necessary to confront and broaden secondary school teachers' knowledge about inductive reasoning to develop their teaching competency. In particular, it would be important for teacher learning and professional development programs to help clarify the use of this reasoning in the mathematical concepts' formation, along with recognising and articulating inductive processes in contexts of mathematical generalization and problem solving. In efforts to enhance understanding of the use of inductive reasoning in teaching, the results of this study could be used to investigate the relationship between the resolution and the use of tasks involving inductive reasoning by secondary teachers and the type of perceptions those teachers have.

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EXPLORING PROSPECTIVE ELEMENTARY MATHEMATICS TEACHERS' KNOWLEDGE: A FOCUS ON FUNCTIONAL THINKING

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Abstract

The importance of students being acquainted with algebraic ideas before secondary education has been revealed in the research literature. It is therefore essential that prospective elementary teachers (PTs) be prepared to instill an early algebra perspective in their teaching. However, PTs often show difficulties in algebra content knowledge, which need to be diagnosed aiming to assist them in developing the required knowledge to teach according to that perspective. This study aims to understand what aspects of functional thinking Spanish and Portuguese elementary PTs exhibit at the beginning of their teacher education program. The findings show that although PTs from both countries use different strategies to generalize functional relationships, the occurrence of successful strategies is low. Also, most participants provide local approaches in their interpretation of relationships between variables and reveal difficulties in understanding and connecting different representations of functions. These difficulties show that PTs lack important knowledge about functional thinking. By providing a framework concerning the functional thinking required for PTs to teach within an early algebra perspective, we shed light on a necessary step for teacher education programs to diagnose PTs' functional thinking and to assist them in developing the needed mathematical knowledge to teach accordingly.

Keywords: Early Algebra, Functional Thinking, Generalization, Prospective Teachers' Knowledge

Abstrak

Pentingnya pengenalan ide-ide aljabar siswa sebelum pendidikan menengah telah terungkap dalam penelitian literatur ini. Oleh karena itu, calon guru sekolah dasar (CG) harus siap untuk menanamkan perspektif aljabar awal dalam pengajaran mereka. Namun, CG sering kali menunjukkan kesulitan dalam pengetahuan konten aljabar, yang perlu didiagnosis dengan tujuan membantu mereka dalam mengembangkan pengetahuan yang diperlukan untuk mengajar sesuai dengan perspektif itu. Penelitian ini bertujuan untuk memahami aspek-aspek berpikir fungsional apa yang diperlihatkan oleh, CG Spanyol dan Portugis di awal program pendidikan guru mereka. Temuan menunjukkan bahwa meskipun CG dari kedua negara menggunakan strategi yang berbeda untuk menggeneralisasi hubungan fungsional, kejadian strategi yang berhasil rendah. Selain itu, sebagian besar peserta memberikan pendekatan lokal dalam interpretasi mereka tentang hubungan antar variabel dan mengungkapkan kesulitan dalam memahami dan menghubungkan representasi fungsi yang berbeda. Kesulitan ini menunjukkan bahwa CG kurang memiliki pengetahuan penting tentang berpikir fungsional. Dengan memberikan kerangka kerja mengenai pemikiran fungsional yang diperlukan untuk CG untuk mengajar dalam perspektif aljabar awal, kami menjelaskan langkah yang diperlukan untuk program pendidikan guru untuk mendiagnosis pemikiran fungsional PT dan untuk membantu mereka dalam mengembangkan pengetahuan matematika yang dibutuhkan untuk mengajar sesuai.

Kata kunci: Aljabar Awal, Pemikiran Fungsional, Generalisasi, Pengetahuan Calon Guru

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In face of well-known problems with the introduction of algebra, usually at ages of 12-13 years-old, there is a growing awareness of the need for engaging students with algebraic ideas earlier, by offering them opportunities to explore intuitive and informal ways of analyzing relationships between quantities, noticing structure in patterns, and studying change, in line with an early algebra perspective (Carraher & Schliemann, 2019; Stephens et al., 2017). In fact, early algebra is becoming part of the mathematics curriculum of elementary grades in different countries (Kieran et al., 2016), establishing the important learning goal of developing students' algebraic thinking as a capacity of making and expressing generalizations (Kaput, 2008). In that perspective we may include the notion of functional thinking, as it represents a form of generalization that involves exploring relationships between quantities that vary together (Blanton & Kaput, 2011).

Teaching in elementary school according to that perspective may represent a great challenge for prospective teachers (PTs) as most of them did not have that kind of experience as students and, therefore, they are not familiar with the algebraic ideas they are required to convey in their future practice (Magiera et al., 2013; McAuliffe & Vermeulen, 2018). However, there is still scarce research illuminating the Specialized Content Knowledge (SCK) (Hill et al., 2008) that elementary PTs effectively need to develop in their preparation to foster students' algebraic thinking, particularly concerning aspects of functional thinking, as well as how teacher education programs may address these issues (Hohensee, 2017; Rodrigues et al., 2019). Moreover, to better document the PTs' functional thinking and difficulties, Lannin et al. (2006) emphasize the need to be attentive to the commonly used frameworks to classify students' approaches to pattern generalization and to adapt them for characterizing PTs' knowledge regarding functional thinking.

This recommendation is particularly important in countries like Portugal and Spain where the elementary mathematics curriculum emphasizes ideas with some resonance with an early algebra perspective, particularly regarding functional thinking, but where there is not a consolidated practice around that perspective in schools (Morales et al., 2018; Oliveira & Mestre, 2014). Carrying out research in two national contexts, with their specificities, may contribute to better evaluate the suitability of a common framework on functional thinking for elementary PTs. Hence, this study aims to understand what aspects of functional thinking Spanish and Portuguese elementary PTs exhibit at the beginning of their undergraduate preparation, when solving algebraic tasks involving functional thinking. Specifically, this study addresses the two following questions: 1) How do PTs generalize functional relationships in patterns? and 2) How do PTs interpret variables and the relationships between them in different representations?

By providing a framework concerning the functional thinking required for PTs to teach within an early algebra perspective this paper intends to shed light on a necessary step for teacher education programs to diagnose PTs' functional thinking and to assist them in developing the needed mathematical knowledge to teach according to that perspective.

Functional Thinking as a Strand of Early Algebra

The important role of functional thinking as a gateway into early algebra, has been emphasized in the context of teaching interventions with elementary students (Carraher & Schliemann, 2019; Stephens et al., 2017). It does not imply a "formal" approach to functions in the sense of introducing and defining function as an object, rather it is a process that involves generalizing relationships between co-varying quantities and expressing those relationships in different representations as well as to use these to interpret and predict function behavior (Blanton & Kaput, 2011; Stephens et al., 2017). Thus, students will be able to build, describe, and reason with and about functions (Blanton & Kaput, 2011).

Generalisation is at the core of functional thinking and has been widely studied in the context of students' exploration of patterns, namely sequences tasks (Kieran et al., 2016; Radford, 2008). This process has received increasing attention in elementary mathematics syllabuses, through the introduction of numerical sequences, presented in different representations such as pictorial or geometric patterns or in numerical tables, as content to be taught, as these describe functional relationships that may support later the study of functions (Apsari et al., 2019; Blanton et al., 2011). In the context of algebraic patterns, the focus of generalization involves ideas such as: (i) grasping a commonality in some cases, (ii) expanding this commonality to all terms; and (iii) discerning a rule or schema to directly obtain any term of the sequence (Radford, 2008).

The literature has evidenced different generalizations strategies when students work with growing sequences, some of them expressing some difficulties, namely applying recursive reasoning that does not allow one to find a general rule or applying incorrectly a whole-object strategy (Moss & McNab, 2011). However, students may regard the difference between consecutive terms not simply in a recursive way, allowing them to determine a direct expression for the sequence. That happens when one uses the common difference as a multiplying factor and makes an adjustment of the result, in the case of non-linear sequences (Barbosa & Vale, 2015). However, sometimes this is introduced to students as a rule without fostering its understanding (Wilkie, 2016).

An important support to students' generalization in sequences is their presentation as a spatial configuration (commonly labeled as 'geometric pattern'). In these situations, students may link the spatial and numerical structures of the sequence, relying upon a visual approach, to realize the commonality in the cases and generalize it to all terms (Radford, 2011).

Conceptualizing and Representing Functional Relationships

In the context of early algebra, two main ways of conceptualizing functional relationships have been considered: covariational thinking and correspondence relationship (Blanton & Kaput, 2011; Confrey & Smith, 1994). According to Thompson and Carlson (2017, p. 423), covariational reasoning means "reasoning about values of two or more quantities varying simultaneously" and has been present in the mathematicians' way of thinking conducting to the modern function definition, although it has not been considered an explicit mathematical concept. Covariational approach may be associated with the notion of coordination of movement between values in the range which means: "being able to move operationally from y_m to y_{m+1} coordinating with movement from x_m to x_{m+1}" (Confrey & Smith, 1994, p. 137). Therefore, in covariational thinking one analyses how two quantities vary simultaneously and thus change becomes a central part of the description of the function (Blanton & Kaput, 2011; Ellis, 2011; Kieran et al., 2016). In the case of numerical sequences, it may lead to the multiple of difference generalization strategy, especially when it is represented in a table.

The correspondence approach to function, the most common in school mathematics, derives from the modern definition of function (Thompson & Carlson, 2017). In opposition with the covariational approach, the correspondence is a static one. Some of the difficulties recognized in meaningful interpretation of functions in opposition to the memorized rules and procedures, that often characterize students' work, may be a consequence of the dominance of this approach. Nevertheless, with the appropriate supportive instruction, even elementary students can describe correspondence in terms of functional rules (Oliveira & Mestre, 2014; Stephens et al., 2017) and thus there are recognized affordances in both covariation and correspondence approaches.

Central to a functional thinking perspective, is how students express the relationships between quantities, represent the associated generalization, and reason with multiples representations such as words, tables, diagrams, graphs, or symbols (Blanton et al., 2011; Kieran et al., 2016; Kusumaningsih et al., 2018), using conventional alphanumeric expressions or idiosyncratic symbols. Several studies show that children can make use of symbolic notation, with understanding, for expressing generalization of functional relationships that are presented in different forms (Blanton et al., 2011; Oliveira & Mestre, 2014). However, an overemphasis on a static view of functions may limit students' ability to generalize those relationships, such as being able to build an equation from a graph depicting a contextual situation by considering how the two involved quantities change simultaneously (Ellis, 2011). Thus, students need to interpret the graphs by local processes, that is focusing point-by-point, but also in a global way, by identifying a trend (Leinhardt et al., 1990). Another difficulty, when students explore reality situations, is that they often perceive the graphical representation as a picture of the physical situation, therefore, they should be given opportunities to explore these different representations so that "graphs become not just visual configurations, but structures embedded with meaning about relationship" (Blanton & Kaput, 2011, p. 16).

Pre-Service Teachers' Knowledge of Functional Thinking

Students' mathematical learning is commonly impacted by teachers' knowledge. Therefore, issues of teachers' content (subject matter) knowledge need to be uncovered and considered both for in-service and pre-service mathematics teacher education (Hill et al., 2008; McAuliffe & Vermeulen, 2018), illuminating how to support PTs to achieve the knowledge they ought to have for developing their future students' functional thinking. In the teachers' knowledge model of Hill et al. (2008), the SCK is seen as a type of content knowledge that enables teachers to "accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures and examine and understand unusual solution methods to problems" (p. 378). Although there are several studies focusing on students' algebraic thinking, the research is still scarce on PTs' algebraic thinking abilities, especially their strategies, misconceptions and difficulties related to the diversity of aspects of functional thinking (Yemen-Karpuzcu et al., 2017).

A few studies targeted PTs' knowledge regarding some algebraic topics related to generalization (including its formulation, representation, and justification), interpretation and use of algebraic symbology, and understanding of functions. Some findings, summarized by Strand and Mills (2014), show that although PTs are often able to generalize numerical and geometric patterns, they tend to have difficulties in interpreting and using efficiently the algebraic symbols. In their strategies to develop algebraic general rules in tasks involving linear, exponential, and quadratic situations, elementary PTs started by drawing and counting to support their thinking and used mainly chunking and recursive strategies (Alajmi, 2016). The challenges that PTs face in generalising explicit rules using symbolic algebraic notation is also frequently documented (Zazkis & Liljedahl, 2002). Even though PTs may express generality using algebraic symbolism, they often struggle in providing justifications for their reasoning what may express a memorization of procedures (Kieboom et al., 2014; Richardson et al., 2009).

Concerning functions understanding, the studies summarized by Strand and Mills (2014) also show that elementary PTs perform well on procedures related to linear functions. However, they have difficulties in interpreting the graphical representations, particularly when one variable is speed, since they confuse it with distance, and in translating between representations (symbolic and visual) and between a representation and its framing context. Still, difficulties in defining a variable and interpreting what it represents, are also documented in the research carried out with PTs (Brown & Bergman, 2013). Moreover, identifying the relationships contained in algebraic expressions and distinguishing between unknowns and variables constitute two aspects of functional thinking that are conceptually challenging to PTs (Hohensee, 2017).

The above studies adopted different frameworks to analyse PTs' knowledge and difficulties concerning specific aspects of functional thinking, such as generalization strategies, the use of algebraic notation to express this generality, and the interpretation of variables and graphical representations. However, as also emphasized by Lannin et al. (2006), to better document or evaluate PTs' proficiency concerning functional thinking it is necessary to develop frameworks covering broader core aspects involved in this process. In this study, to investigate PTs' functional thinking regarding the strategies to generalize functional relationships, the interpretation of variables and relations between them, and the connections between representations, we developed a framework, described in the next section, that encompasses and articulates different aspects that have been discussed above.

METHOD

Participants

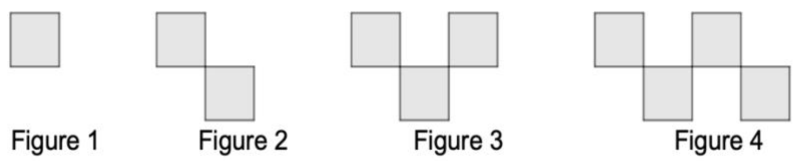
The participants in this study are 94 (35 male and 59 female) Spanish and 70 (2 male and 68 female) Portuguese PTs who attended the 1st year of a degree in elementary education teaching, respectively at a public university in North of Spain and at two public high schools of education (labeled by A and B) in the center of Portugal. These institutions were chosen as a purposeful sample among those where we knew PTs had not received any specific teaching on algebraic thinking and teacher educators agreed to collaborate. All the PTs volunteered to participate in the study and had little or no exposure to early algebra previously in their elementary education.

Instrument and Data Collection

A questionnaire to assess PTs' algebraic thinking was developed by the researchers (authors). The six tasks that integrate the questionnaire were selected from the literature and adapted by modifying their statements and including new items. Then, the questionnaire was evaluated by eight specialists and trialed with pilot samples of elementary PTs from both countries (not participants in the study). These outcomes were discussed by the authors and further adaptations to the questionnaire were made to refine the structure and wording of the items in the tasks. The questionnaire was then applied in both countries at the beginning of the school year. In this paper, we focus on data collected from participant PTs' answers to three tasks from the questionnaire that involve functional thinking.

The "Geometric pattern" task (Figure 1), used to answer the research question 1, asks for a generalization of a geometric growing pattern (linear), allowing PTs to use diverse strategies to establish relationships between quantities and to describe and represent those relationships using multiple representations.

Look at the following figures:



If we consider a *vertex* to be the point where two or more segments meet, we have that Figure 1 has 4 vertices; Figure 2 has 7 vertices; etc.

Q1. Explain, using two different approaches, how many vertices the figure consisting of 25 squares will have.

Q2. Represent the relationship between the number of vertices and the number of squares in any figure of the sequence.

Figure 1. Geometric Pattern Task (adapted from Blanton et al., 2011)

The "Representations" task (Figure 2) intended to assess aspects of PTs' functional thinking such as: interpretation of variables and of relations between variables using co-variation and correspondence approaches to generalization; and connections between representations to interpret relationships.

Here are three solutions to the following problem: "Find the real values *a* and *b* that hold *a*+2=*b*+10". Indicate which solution (or solutions) you consider correct, and which incorrect. Justify your answer.

Solution 1:

"The solution are the numbers in the table"

a	b
8	0
9	1
10	2
11	3
12	4
13	5

Solution 2:

"The solution is a=10 and b=2 because 10+2=2+10"

Solution 3:

"The solution are the points on the line r"

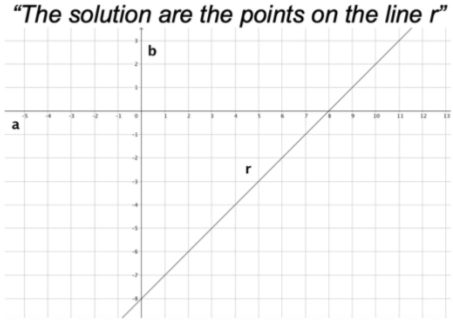
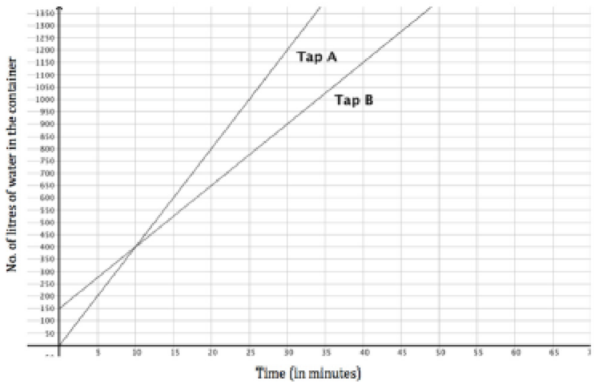


Figure 2. Representations Task (adapted from Hart, 1981)

Similarly, "Deposits" task (Figure 3) concerns the PTs' interpretation of variables and connections of different representations to interpret functional relationships, using covariation or correspondence approaches that could be local or global. So, both tasks were used to answer the research question 2.

To fill two water containers (container A and container B), both with a capacity of 1000 liters, two taps were opened (tap A and tap B respectively), with constant flow. The following graph shows the number of litres in each container over time (in minutes) since the opening of the taps. Provide a justified answer to the following questions:



Q1. Describe what happens with the containers at time 0.

Q2. What does the intersection point of the two straight lines of the graph relative to taps A and B represent?

Q3. What is the deposit that reaches its maximum capacity more quickly? After how long since the opening of the tap?

Figure 3. Deposits Task (adapted from Branco, 2013)

Data Analysis

This qualitative study followed a descriptive and interpretative analysis (Erickson, 1986) of PTs' solutions of the tasks, including a description of quantitative data organized in frequency tables. The pre-established categories we considered to interpret the PTs' functional thinking and the difficulties emerging from their work (Table 1 and Table 2) come from prior research on students' functional thinking, as described in the previous sections, particularly those from Barbosa and Vale (2015) and Leinhardt et al. (1990), with adaptations in the language used according to Ayalon et al. (2016). We include for each category a description of possible approaches which are task-specific, to generate a full picture of PTs' answers. Altogether the categories concerning generalizing functional relations and interpretation of variables and relationships provide a framework to characterize PTs' knowledge regarding functional thinking within an early algebra perspective.

To identify possible strategies to generalize a functional relation (both for distant generalization and a general term) in "Geometric pattern" task, we considered four categories (Table 1).

Table 1. Categories for Coding Strategies to Generalize Functional Relations

Category	Description: The PTs…
Counting	draw the next figures and count their elements
Difference	
Recursive	continue the sequence using the numerical difference between consecutive terms or explicit the recursive relation between consecutive terms.
Multiple of difference	use the difference between consecutives terms as a multiplicative factor (adjusting or not the result) to obtain distant terms or the general term.
Multiplicative reasoning	
Missing value	use the rule of 3 to find a distant term.
Proportional	use multiplicative strategies, starting from one known term of the sequence to find distant terms or the general term.
Correspondence	
Visual	express a relation between the two varying quantities for a distant term or in the general term, based on the characteristics of the pictorial representation.
Numerical	express a relation between the two varying quantities for a distant term or in the general term, based on the numerical sequence.

The categories used in "Representations" and in "Deposits" tasks (Table 2), attempt to capture the ways in which PTs interpret variables and relationships between them, including the connections between representations they establish. Interpretation in these categories means the action by which a PT gains meaning from a graph, equation, or context (Leinhardt et al., 1990).

The PTs' answers were independently coded by the authors, focusing on the identification of the categories proposed. To assure the validity of the analysis and to increase the reliability of the results, a check-scoring of PTs' answers of a random selection of 10 Spanish and 10 Portuguese initially coded questionnaires was undertaken by the authors to reach consensus. An inter-rater reliability was calculated for this sample of data covering all questions of the tasks and in all the codes an agreement of at least 82% was found among authors, which is considered satisfactory. Divergent interpretations or doubts concerning a codification were discussed until full agreement was reached.

Table 2. Categories for Coding Interpretation of Variables and Relationships

Category	Description: The PTs…
Interpretation of variables	interpret the variable as a *varying quantity* or as *unknown* in an equation.
Interpretation of relationships between variables	
Covariation	coordinate the two variables mentioning how dependent and independent variables change simultaneously rather than mentioning them separately, connecting different representations to identify this relation.
Global Correspondence	identify and/or explain the direct relation between two variables, connecting different representations to identify this relation, focusing on patterns, and gaining meaning about the relationship between variables.
Local Correspondence	identify and/or explain the direct relation between two variables, connecting different representations to identify this relation, determining when specific events or conditions are met.
Connection between representations	connect two representations to interpret a third one.

In the next section, we present in tables a quantitative descriptive analysis of the PTs' answers to each of the tasks and illustrate our interpretation of the ideas associated to each category with detailed examples of their answers (mentioned as S# in case of Spanish PTs and PA# or PB# for Portuguese PTs to assure their anonymity). The examples may provide evidence of PTs' functional thinking or the difficulties they reveal in their work.

RESULTS AND DISCUSSION

Geometric Pattern Task

The incidence of the different strategies that were used by the PTs from both countries are presented in Table 3. In some cases, PTs provided two different approaches in each of the questions (the distant term and the general term of the sequence), which were all considered in the analysis. For the distant term question, a total of 88 different answers (A) for Portuguese PTs and 125 for the Spanish were registered, as they were asked to present two different strategies. Regarding the general term, we found 36 answers for the Portuguese PTs and 69 for the Spanish ones. Next, we discuss these results separately for the distant term and the general term.

Table 3. Strategies Used by PTs for Finding the Distant Term and a General Term

Category		Distant term		General Term	
		Portuguese (A= 88)	Spanish (A=125)	Portuguese (A= 36)	Spanish (A=69)
Counting		15 (6)	14 (3)	–	–
Difference	Recursive	33 (25)	11 (2)	47 (0)	35 (0)
	Multiple of difference	19 (15)	6 (6)	25 (22)	7 (7)
Multiplicative reasoning	Missing value	8 (0)	20 (0)	0 (0)	7 (0)
	Proportional	1 (0)	1 (0)	8 (0)	0 (0)
Correspondence	Visual	18 (16)	40 (30)	11 (8)	35 (26)
	Numerical	1 (1)	2 (2)	3 (0)	1 (1)
Uncategorized		5 (0)	6 (0)	6 (0)	15 (0)
No answer (% of the PTs)		17	9	50	29

Note. The first number on each cell gives the percentage for each strategy, and between brackets the percentage of correct answers.

PTs' Strategies for Finding the Distant Term

Portuguese and Spanish PTs were able to provide a correct answer to find the distant term in 63% and 43% of the cases, respectively. The correspondence strategy was the preferred strategy to find the distant term for the Spanish PTs, which is present in 42% of their responses (32% correct), mostly through a visual approach. An example of a correct use of such strategy is the following PT's answer that expresses a relation between the two varying quantities for a distant term as referring to the characteristics of the pictorial representation: "2 squares lose one vertex, 3 squares lose two vertices, 4 squares lose three vertices, 25 squares lose 24 vertices. 25x4=100, 100-24=76" (S6). We also find incorrect implementations of this strategy, where the PT misses the extra vertex from the first square: "It will have 75 vertices since when you put 2 squares together, they share a vertex, so 25 squares x 3 vertices equals 75 vertices in total" (S10).

The difference-recursive strategy was most frequent among the Portuguese PTs (33% of their answers), leading them to a correct answer in most cases (25% of the total), as the example in Figure 4. We can observe that the PT begins by considering the number of vertices of the figure formed by 5 squares (16 vertices) and keeps adding 3 until reaching the figure with 25 squares, concluding: "The figure formed by 25 squares will have 76 vertices".

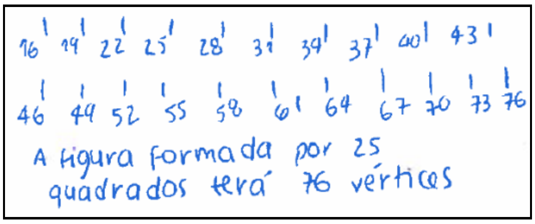


Figure 4. Example of Difference-Recursive Strategy - PB24

The difference strategy of type multiple of difference was the second most used (19% of responses) by Portuguese PTs to find the distant term. After realizing that there is a constant difference between the terms, these PTs used it as a multiplicative factor to determine the required distant term, making an adjustment at the end by adding the extra vertex of the first square. The example in Figure 5 shows that the PT realizes that the 25th term is obtained by multiplying 24 by 3 and then adjusting the result by adding up four from the first term. It is also worth noting that most Portuguese PTs who used this strategy obtained the general term already in their answer to this question, and applied it to correctly find the distant term, as the example in Figure 5.

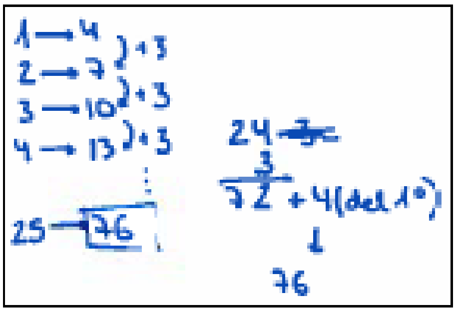


Figure 5. Example of Multiple of Difference Strategy - PA19

Multiplicative reasoning strategy of type missing value was used in 20% of Spanish PTs' answers to find the distant term. This strategy occurs when the participant uses the rule of three to find a distant term, which does not result in a correct solution since this is not a proportional numerical relation. For instance, one PT states: "By a rule of three: 1 square – 4 vertices; 25 squares – x. [So] x=100 vertices" (Figure 6). This strategy was not frequent among Portuguese PTs.

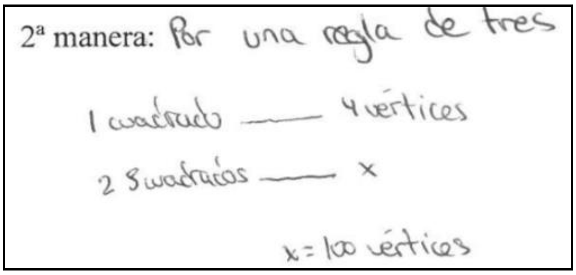
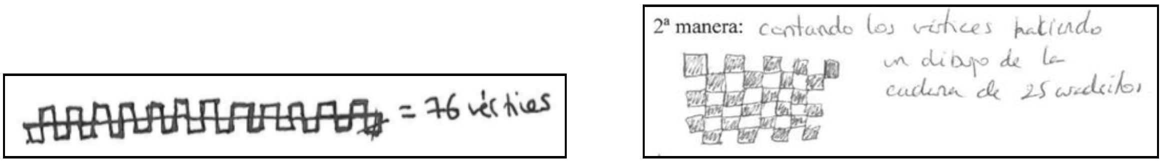


Figure 6. Example of Incorrect Missing Value Strategy - S8

The counting strategy based directly on the figural pattern, it is still present in about 15% of the answers, by both Portuguese and Spanish PTs. In most cases, this strategy led them to incorrect answers as PTs were not able to understand how the pattern grows. Two examples of Spanish PTs' answers, one correct (Figure 7(a)) and one incorrect (Figure 7(b)), are given below. For example, S77 (Figure 7(b)) explains: "By counting the vertices of a drawing of the 25-squares chain". Finally, we should remark that 17% of Portuguese PTs and 9% of the Spanish PTs did not provide an answer to the distant term question.



(a) (b)

Figure 7. Examples of Counting Strategies - S7(a) and S77(b)

PTs' Strategies for Finding the General Term

Many PTs from both countries were not able to provide the general term: 50% of Portuguese and 29% of Spanish PTs did not answer this question. Only 30% of the Portuguese PTs and 34% of the Spanish who answered this question provided a correct general term for the sequence. For those cases, a description on how they represent the relationship (using words, syncopated language, or mathematical symbols) is presented.

We find that most Portuguese PTs' strategies (47%) were difference-recursive type, not leading to the general term, as also frequent among Spanish PTs' answers (35% of the strategies), like the one of S33 (Figure 8), who explains: "Every time we add a square, one of its vertices is shared with the previous, adding 3 vertices per square instead of 4, 1 less than in the case of the first".

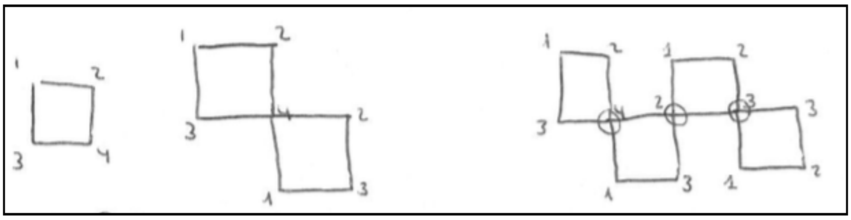


Figure 8. Example of a Difference-Recursive Strategy - S33

Portuguese PTs most successful strategy for the general term was the multiple of difference strategy (25% of the strategies), which led them to the correct answer in most cases (22% in total). As mentioned before, most of the PTs found a general term already when answering the question for the sequence's distant term. Thus, it seems that the PTs were using a procedure they have learnt to determine the general term of a sequence representing an affine relation between variables. In Figure 9, we can notice that the PT stresses the difference between consecutive terms ("+3"), and, using this difference as a multiplicative factor, writes down the general term in symbolic mathematical language. Then, the PT applies the relation to find the distant term.

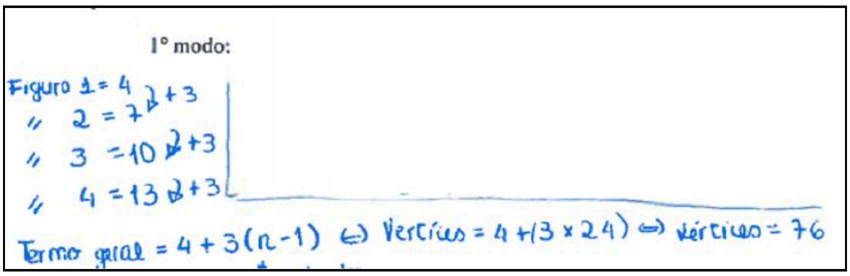


Figure 9. Example of Multiple of Difference Strategy - PB14

The Spanish PTs' most common strategy to find the general term was the correspondence strategy (35% of the strategies). These PTs used a visual approach in most cases, such as the following one: "Number of vertices [in each square] times the number of squares and then you subtract the number of squares minus 1" (S53). Here the PT expresses the direct relation between the two varying quantities in a general rule using words and referring to the contextual features of the sequence (squares and vertices). Therefore, the way the relationship between the variables is expressed, which could be written as 4×n-(n-1), is dependent on how the PT has apprehended the structure of the spacial sequence.

It is also worth noting that very few PTs, among those who found the functional relationship, provided a full symbolic equation to express it, such as the one presented by S62 who wrote: "3n+1=v, n= nr squares, v=vertices".

Overall, these results showed that PTs from both countries encountered serious difficulties when trying to find the general term for the two varying quantities and adopted diverse approaches in the generalization of patterns: Portuguese PTs privileged the multiple of difference strategy, whereas the Spanish preferred a correspondence visual strategy.

Representations Task

The results concerning the approaches used by the PTs to interpret variables and relationships between them, including the connections between representations they established, were identified on the obtained PTs' answers, and are presented in Table 4. In the following sections, we discuss the three categories separately.

Table 4. PTs' Interpretation of Variables and Relationships between Variables and Connection between Representations

Category		Portuguese (N=70)	Spanish (N= 94)
Interpretation of variables	As a varying quantity	27 (11)	72 (11)
	As two specific unknowns	10 (0)	9 (0)
	No evidence of interpretation	20	10
Interpretation of relationships between variables	Global correspondence	13 (11)	28 (11)
	Local correspondence	21 (0)	44 (0)
	No evidence of interpretation	23	19
Connection between representations	Connect two representations to interpret a third one	33(11)	46 (11)
	No evidence of connection	24	45
No Answer (% of the PTs)		43	9

Note. The first number on each cell gives the percentage of answers for each category, and between brackets the percentage of correct ones. "No Answer" concerns the percentage of the PTs that did not answer to this task, thus in each category it completes the 100% of answers.

Interpretation of Variables

We can observe that 72% of Spanish PTs interpreted variables as a varying quantity (Table 4) but only 27% of Portuguese PTs showed such interpretation. An example evidencing this interpretation is the answer of a Spanish PT, who argues: “The correct solution is the third one, since for each value of a there is a value of b, with all possible options, including negative numbers, and the previous [solutions] are more limited since they only provide some solutions” (S33). This PT’s answer shows an understanding that the two letters may assume many diverse values, as the solution offered by the graph, also providing a justification. Other PTs express the same interpretation of variables, referring to the diversity of values they may assume, but their responses were classified as incorrect since they consider the solution provided in the table as the correct one, as the following example of a Portuguese PT shows: “The correct solution is Solution 1 since it provides a sequence of possible options. Solution 2 is the incorrect solution since it only has one solution and there can be more” (PA13). These incorrect answers may result from the PTs’ difficulties in interpreting the graphic representation and thus not considering it as a possible solution.

We also find that about 10% of both Portuguese and Spanish PTs interpreted the variables as two specific unknowns. This was demonstrated for instance by a PT who explains: “Solution 2 is correct because if a is 10, added to 2 gives 12, and if b is 2, added to 10 gives 12. They are symmetric. Solution 1 is wrong because there is only one situation in this table that meets a+2=b+10” (PA1). We interpret that the PT believes the relation between quantities to be adding up to 12, and that holds only for a specific value of each letter. Also, worth noticing is the incorrectness of the language used by the PT when stating that the numbers are “symmetric”.

Finally, we observe that 20% of Portuguese PTs and 10% of the Spanish PTs showed no evidence of interpretation of the variables. In these cases, the PTs did not provide an explanation for their choice or seemed not to have understood the question.

Interpretation of Relationships between Variables

In this category, it was analysed how PTs interpreted the relationship between variables, distinguishing global correspondence approaches from local ones. We observed that 21% and 44% of the Portuguese and Spanish PTs, respectively, showed a local correspondence approach in their responses. This has been classified as incorrect as it does not consider that the relation between the values applies to an infinite set. We find an example, classified as incorrect, in the following answer: “The correct solution is solution 1 [table]. This is the one that includes all the values that make the expression a+2=b+10 true” (PA8). Here some of the values provided in the solutions are considered by the PT to hold the expression a+2=b+10, but there are again no indications to suggest that more than a finite set has been considered.

We find evidence of a global correspondence approach in 13% and 28% of the Portuguese and Spanish PTs, respectively. An example of a global approach is found in the following answer that refers to the infinity of the set of solutions: "The only one representing all solutions is solution 3, which defines well the R-representation of a line that has neither a beginning nor an end" (PA3). Some PTs also evidenced a global correspondence approach, but they did not consider the solution provided by the graph as the only one that is correct, as the case of a PT who only labels "Solution 1" as correct and explains that: "Any number a that exceeds in 8 any other number b corresponds to a+2=b+10 since 10 exceeds 2 by 8" (S2). They consider how both variables change in relationship to one another, and to the existence of many more values than those on the table, but not in the graph.

There are still 23% and 19% of the Portuguese and Spanish PTs, respectively, whose answers showed no evidence of interpretation. An example providing no evidence is the one by a PT who argues: [The first solution in the table] is not correct since zero cannot be in an equality" (S19), showing a lack of knowledge regarding algebraic expressions.

Connections between Representations

Only 11% of both Portuguese and Spanish PTs' answers showed, correctly, connection between representations. In these answers, the PTs understand the algebraic expression provided in the task and connected it with the table and the graph representations, emphasizing that the solution given by the graph is the only complete one, as showed in the following example: "I consider all solutions to be correct even though I consider solution 3 totally complete, since solution 1 and 2 provide examples of solutions, and solution 3 gives a more comprehensive solution" (PB7). Other answers, although showing connection between representations, were classified as incorrect since the PTs failed in not considering the graph as the only one that displays all solutions of the algebraic expression, like the following: "Solution 1 is correct since giving values to a or b, those numbers come out. Solution 3 is correct since it is a representation of the table of Solution 1" (S26).

The answers classified as no evidence of connection between representations (24% and 45% of Portuguese and Spanish PTs, respectively) mostly show a lack of understanding of the graphic representation. That is the case of a PT who explicitly states: "Solution 3 cannot be right since the right line r has nothing to do with a and b" (S47). Another example is given by a Spanish PT who argues about the graph in solution 3 as follows: "Incorrect: if you place the numbers of the line by the given equation, you obtain: a=8, b=-8; 8+2=-8+10, 10=2, which is not true" (S45), and another by a Portuguese PT who explains "This solution [solution 3] is not correct since -8+10=2 and 8+2=10. So, $-8+10 \neq 8+2$" (PA2). It is noticeable here that the two PTs are using the same incorrect argument when rejecting Solution 3. They wrongly consider the point (-8,8) to be a point on the straight line and argue that the line cannot be a solution to the problem since that point does not hold the expression.

In summary, the results concerning the Representations task reflect many difficulties among PTs of both countries in all categories of the analysis. The Portuguese PTs had a lower level of participation than Spanish ones, in general, but a very low percentage of correct answers among PTs from both countries were observed. This brings to light important PTs' difficulties regarding the understanding of algebraic expressions and their representations.

Deposits Task

The results of the approaches used by the PTs to interpret relationships between variables given by a graph, identified on their answers to the three questions in this task, are presented in Table 5.

Table 5. PTs' Interpretation of Relationships between Variables

	Portuguese (N= 70)			Spanish (N=94)		
Interpretation of relationships	Q1	Q2	Q3	Q1	Q2	Q3
Co-variation	—	—	0 (0)	—	—	2 (1)
Correspondence	58 (56)	61 (49)	52 (26)	93 (81)	94 (83)	86 (40)
No evidence of interpretation	6	6	17	4	3	9
No Answer (% of the PTs)	36	33	31	3	3	3

Note. The first number on each cell gives the percentage of answers for each category, and between brackets the percentage of correct ones.

In the case of Q1, 56% and 81% of Portuguese and Spanish PTs, respectively, provided a correct answer (deposit A: empty, and deposit B: 150 liters). It was interpreted herewith that all PTs who were able to identify variables and their relationship used a local correspondence approach. Some answers were incorrect, like the following one: "Tap A is slower so at the instant 0 there are 0 liters, while Tap B comes out faster because at the same time there are already 150 liters" (S45). The PT can identify the coordinates, but he focuses on the taps' speed, which reflects difficulties in interpreting the variables. Other answers to Q1 were classified as no evidence of interpretation of relationships, like: "Tap A had nothing in its flow" (S22) or "Without water" (PA22). We take that the absence of explanations in these answers reflects a lack of understanding of relationships between variables.

Regarding Q2, we find that most PTs who provided an answer showed a correspondence approach (49% and 83% of the total of Portuguese and Spanish PTs respectively), like one Portuguese who argues "Within 10 minutes both deposits have 400 litres" (PB18), showing therewith to interpret the meaning of the variables and their relation from the graphic representation. About 12% of both the Spanish and Portuguese PTs showed correspondence approach in Q2 but their answers were considered incorrect since they established a relation between the time and the number of litres in the deposits but failed to explicitly refer to the variables, as observed in the following answer: "That at that time they had the same capacity" (S6). This PT uses the word "capacity" to mean 'fill level', a mistake quite frequent among Spanish PTs. It was also very common among PTs from both countries to allude erroneously to the taps in their justification instead to the deposits.

Few responses to Q2 were classified as providing no evidence of interpretation of relationships like in the following argument: "The moment in which both taps get the same flow and, from there, they have different rhythm" (S36). The PT shows not to understand the dependent variable (and therefore the relationship between the two variables) since he assumes that it represents the flow instead of the amount of water.

Both co-variation and correspondence approaches in the interpretation of relationships between variables were evaluated by Q3 where PTs were asked: (1) to name the deposit that reaches its capacity faster, and (2) the time when that occurred. Most PTs that provided an answer adopted a correspondence approach (52% and 86% of Portuguese and Spanish PTs, respectively). About half of these answers were correct, like the one by a Spanish PT who argues: "The deposit A is the one that reaches its maximum capacity 1000 litres more quickly, at 25 minutes, versus deposit B that reaches it 10 minutes later" (S38). The acknowledged correspondence between the variables is evident in some cases by the marks in the graph to connect the values of the coordinates of each point (Figure 10).

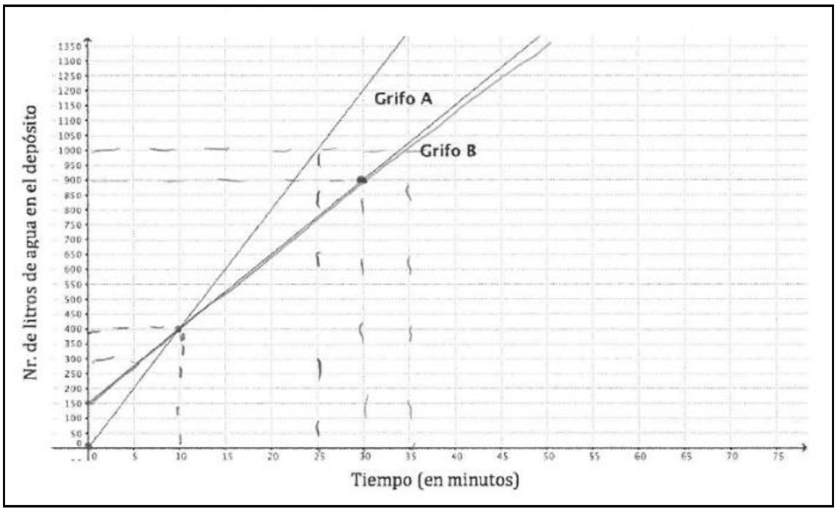


Figure 10. Example of Correspondence in the Graph - S38

Many of the mistakes occurred when the PTs considered the capacity of the deposits to be the top of the ordinate axis in the graph (1350 litres), and not the 1000 litres stated in the task, as shown in the answer: "A takes 35 minutes and B 50 minutes approximately" (S45). Just two Spanish PTs showed a co-variation approach. One of them argues that: "Tap A since it has a greater slope" (S17).

Few PTs provided incorrect answers when answering Q3 which were classified as no evidence of interpretation of relationship, such as this one: "Tap A has more power, so whatever its start, it fills first" (S22). It was common among Spanish PTs to make assumptions about contextual aspects that were not provided by the information in the graph, or by the task's statement.

This empirical study examined aspects of Spanish and Portuguese elementary PTs' functional thinking at the beginning of their undergraduate preparation, as they solved algebraic tasks. The results show that PTs from both countries have difficulties in generalizing algebraic rules and in interpreting variables and the relationship between them, as in other studies (e.g. Alajmi, 2016;

Hohensee, 2017). Nevertheless, when compared to previous research, this study gives a broader perspective about elementary PTs' functional thinking as it encompasses more core dimensions.

As elementary PTs need to develop a specific SCK for teaching in an early algebra perspective, being able to identify the relationship between the two variables in a sequence is of paramount importance. Addressing the first research question in this study, we found that among the Portuguese PTs there is a prevalence of recursive-difference strategies, both for determining the distant term and the general term of the sequence that may not comprehend a functional relationship. This approach may result from a memorization of a procedure, which is often identified in other studies (Kieboom et al., 2014; Wilkie, 2016). On contrary, the Spanish PTs who were able to determine the distant or the general term tend to use a correspondence approach, relying on the visual characteristics of the pattern, but the occurrence of successful strategies is still low among them. It is also worth noting that very few PTs from both countries used a full symbolic equation for the general term which may be an indicator of their difficult with the algebraic language, as stressed by Strand and Mills (2014).

In what concerns the second research question, specifically with a focus on the PTs' understanding of variables, we found that Spanish PTs considered variables as a varying quantity in a higher percentage than Portuguese PTs but had equally a low rate of correct responses. PTs from both countries provided local approaches in their interpretations of relationships between variables more often than global ones (Leinhardt et al., 1990). Particularly, Spanish PTs showed this approach very often by checking whether the values of the table fulfilled the relationship given by the algebraic expression. However, the tendency to see the table as a finite set of coordinated points, attached to a local view of function, may be linked to a static view of function, preventing PTs from searching how the values of the two variables change simultaneously (Ellis, 2011). In general PTs also reflect many difficulties in understanding the algebraic expression presented in the task's statement, and therefore hardly connect it with other representations, as also pointed out in other studies (Strand & Mills, 2014). In face of these two difficulties altogether it is not surprising that the majority of PTs from both countries have shown a lack of important knowledge regarding the graphical representation of a function, namely that it conveys all the range of coordinates for the function. When interpreting the relationship between variables that describe quantities from a real context represented in a graph, there is an increased number of PTs who can describe the functional relationship at a local level. Nevertheless, there is still an important number of Portuguese PTs who do not provide an answer and a majority from both countries who reveal difficulties in connecting the information about the variables provided by the graph with the real context. This finding suggests that using graphs to model concrete situations may not only entail great complexity for students (Patterson & McGraw, 2018), but also for many elementary PTs, and it may prevent them from promoting the exploration of such representations in their future teaching practice.

CONCLUSIONS

Although PTs can use different strategies to generalize functional relationships, the occurrence of successful strategies is low. Also, most participants provide local approaches in their interpretation of relationships between variables and reveal difficulties in understanding and connecting different representations of functions. These findings show that regardless of PTs' school experiences in mathematics and the differences among the curriculum in both countries, PTs lack important knowledge about functional thinking when they start their preparation to become elementary teachers. From this study, we can derive some implications for teacher education.

First, considering that elementary PTs may have quite diverse mathematical backgrounds, teacher educators need to understand the key mathematical ideas they have developed. A framework like the one proposed in this study may be a starting point to identify different aspects of PTs' functional thinking to understand their misconceptions and difficulties with algebraic ideas, as recommended by Yemen-Karpuzcu et al. (2017), as well as if they tend to rely on the use of rules and procedures, without understanding, when solving tasks that involve functional relationships. Second, exploring core ideas on functional thinking can also be used as a rich context for PTs to rebuild the mathematics they have learnt, namely by stablishing connection among topics they often see in isolation, as it happens sometimes with sequences and functions. An attention on PTs' functional thinking in an early algebra perspective is also essential for their understanding of the mathematics behind the tasks to propose to their future students.

Finally, and more specifically, opportunities should be given to PTs to: (i) reflect on the different strategies and their level of efficiency for generalization of functional relationships and (ii) deepen their understanding of functional relationships in different representations and how they connect with each other. In what concerns the first aspect (i), we may illustrate that the use of a difference-recursive strategy that has been mentioned in the literature has not led students to understand the structure of the patterns, nor the relationship between the quantities involved. However, using a multiple of difference approach can be a successful strategy when students understand the relation between the numerical sequence of values and the general term. At the same time, we want to support PTs' functional thinking, encouraging them to advance to more sophisticated strategies based on correspondence approaches, but still, as future teachers, they need to understand that simpler strategies are also important and can provide opportunities for students' further development. As most PTs did not experience this kind of activities as students, teacher education programs should provide opportunities for them to explore generalization in different contexts.

Concerning the second aspect (ii), teacher educators should create situations that allow PTs to understand how different representations may support thinking in functional terms, either in a covariational or correspondence approach. With the strong incidence in a static view of function, associated with the correspondence approach, throughout the middle and secondary schools (Ellis, 2011), elementary PTs may need to further explore tables and graphs as means to understand the relationship between the variables of a function, from a global perspective, embedded in significant contexts for elementary students.

Future research may address how the framework adopted in this study may support the design of teacher education programs for promoting PTs' functional thinking, regarded as an important dimension of SCK for elementary teachers, as well as other dimensions of PTs professional knowledge, namely the knowledge about their future students.

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GROWTH MINDSET, SCHOOL CONTEXT, AND MATHEMATICS ACHIEVEMENT IN INDONESIA: A MULTILEVEL MODEL

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Abstract

Shifting students to a growth mindset can increase their achievements. Nevertheless, only a few studies have been conducted on this topic in developing countries. This study aims to examine the relationship between growth mindset, school context, and mathematics achievement in Indonesia. Using a multilevel model on the PISA 2018 data, this study explored the variables that contributed to mathematics achievement. The multilevel analysis showed that students' gender, growth mindset, index of economic social, and cultural status were statistically significant predictors of students' mathematics achievement. Girls have been reported to have a higher mathematics achievement than boys in Indonesia. As the students' growth mindset increases, so do their mathematics achievement.

Keywords: PISA 2018, Mathematics, Multilevel, Growth Mindset

Abstrak

Keyakinan siswa terhadap *growth mindset* atau pemikiran yang berkembang dapat meningkatkan prestasi belajar mereka. Namun hanya beberapa penelitian yang telah dilakukan pada topik ini di negara berkembang. Penelitian ini bertujuan menyelidiki hubungan antara *growth mindset*, konteks sekolah, dan prestasi belajar matematika siswa di Indonesia. Menggunakan model multilevel pada data PISA 2018, penelitian ini mengeksplorasi variabel-variabel yang berkontribusi pada prestasi belajar matematika. Analisis multilevel menunjukkan bahwa jenis kelamin siswa, *growth mindset*, indeks sosial ekonomik dan status budaya merupakan prediktor yang signifikan secara statistik terhadap prestasi belajar matematika siswa. Pelajar perempuan dilaporkan memiliki prestasi matematika yang lebih tinggi dibandingkan pelajar laki-laki di Indonesia. Seiring meningkatnya *growth mindset*, prestasi matematika siswa juga mengalami peningkatan.

Kata kunci: PISA 2018, Matematika, Multilevel, Pemikiran Berkembang

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In December 2019, the Organization of Economic Cooperation and Development (OECD) published the result of the Programme for International Student Assessment (PISA) 2018, which placed Indonesia in the quadrant of low performance and high equity (Avvisati et al., 2019). The average mathematical score of Indonesian students were 379, far below the average mathematical score of all PISA participants which equaled to 489. In the first participation in the year 2000, Indonesian students had obtained an average of 367 for mathematics score. From 2003 to 2015, the average mathematics score fluctuated between 360 and 386 (Nugrahanto & Zuchdi, 2019). As a consequence, large efforts should be done to improve students' mathematical ability.

The PISA is a routine evaluation or assessment held by the Organization of Economic Cooperation and Development (OECD) to measure the literacy of students around the world (Stacey, 2011). Besides the scores obtained from standardized tests, some additional data are collected from students, parents, and teachers using questionnaires. Consequently, the result of the PISA test can be compared at various levels and further analyzed to find a way for improving the educational quality.

For a long time, some Indonesian as well as foreign researchers were trying to explain the PISA results using their datasets. Thien et al. (2015) used multilevel analysis on PISA datasets to compare the mathematics performances of students from Indonesia, Malaysia, and Thailand. This study found that attitude towards learning outcomes and mathematics self-efficacy were the main factors affecting student level performance in Indonesia. Pakpahan (2016) used correlation analysis on the PISA 2012 data and found that students' discipline, socioeconomic and cultural conditions, computer ownership, and textbooks were the main factors influencing the achievement of Indonesian students' mathematical literacy. Kartianom and Ndayizeye (2017) used the multilevel model to analyze the PISA 2015 data and found that the socio-economic status of the family, the socio-economic average of the school, and students' sense of belonging to mathematics affect the Indonesian students' mathematics achievement. In Serbia, a multi-level analysis revealed that students' achievement in mathematics is affected by gender, non-cognitive characteristics, habits of study, student-perceived teaching quality, and several school-level factor (Lazarević & Orlić, 2018). Using a binary multilevel model to analyze the PISA 2012 data, Karakolidis et al. (2016) found that students' gender, immigration status, self-constructs in mathematics, and the mean socio-economic status (SES) in school significantly affect the students' mathematical achievement in Greece. Using the hierarchical linear model, Anderson et al. (2009) analyzed the students' mathematical literacy based on the PISA 2006 data.

In PISA 2018 test, a new important variable has been added into the student questionnaire, namely growth mindset. A growth mindset is the belief that someone's ability and intelligence can develop over time (Caniëls et al., 2018). Blackwell et al. (2007) as well as Dweck (2007) show that students with a growth mindset are more likely to believe that learning and understanding require efforts. When faced with challenges, they may be more willing to make more effort and take more risk. A growth mindset is also related to poverty, where more students from a lower-income family exhibit a fixed mindset (Claro et al., 2016). Change in students' mindsets can be affected by academic experiences, peers, and formal learning (Limeri et al., 2020).

Related to the mathematics ability, students in all levels may explore mathematics if they know that mathematics can be learned (Alpar & Van Hoeve, 2019). However, the belief that ability is a fixed trait (instead a growth trait) is particularly common and may be a key reason for students' underperformance and disinterest in mathematics (Sun, 2018). In United States, it has been proven that a short intervention in growth mindset can improve grades among lower-achieving students and increased overall enrolment to advanced mathematics courses in secondary education (Good et al., 2012; Yeager et al., 2019). Similarly, a fixed mindset may contribute to poor student performance, inequitable participation, and disinterest in mathematics (Horn, 2007; Boaler et al., 2018). However, a large meta-analysis by Sisk et al. (2018) shows that the overall effect of growth mindset on academic achievement was weak. Despite these differences, the growth mindset still becomes a popular research topic around the world (Sun, 2019).

The PISA 2018 result shows that the majority of students across OECD countries has a growth mindset, as evident by their responses ("disagree" or "strongly disagree") with the statement "Your intelligence is something about you that you can't change very much". In contrast, at least 60% of students in Indonesia believe that their intelligence is something that they cannot change by themselves, which represents the fixed mindset (Avvisati et al., 2019). Similar results have been observed in students from Dominican Republic, Kosovo, Panama and the Philippines, which are countries with low achievement in the PISA 2018 test.

The dominance of fixed mindsets in Indonesian students leads us to explore the PISA 2018 data and examine whether a growth mindset contributes significantly to their mathematics achievement. To ensure the effects, we examine several variables that significantly affect students' mathematics achievement. Following some earlier studies using PISA datasets (e.g. Kartianom & Ndayizeye, 2017), this study uses the multi-level model framework to see the effects on both the student and school levels.

METHOD

Data Collection

The data set used in this study are the PISA 2018 data. The PISA is a triennial survey of 15-year-old students assessing to what extent they have acquired the key knowledge and skills that are essential for full participation in society. The assessment focuses on proficiency in reading, mathematics, science and an innovative domain, and as well as on students' well-being (OECD, 2019). The Indonesian PISA 2018 data set includes all observations from 12,098 students and 397 schools. The response variable is students' mathematics achievement which is calculated by averaging ten plausible values of the mathematics scores. The structured data were found in PISA 2018 in Indonesia, where the level-1 of hierarchy is the students and the level-2 is the school. The level-1 predictors in this study are students' gender, students' growth mindset, and students' socio-economic status which is estimated by the PISA index of economic, social, and cultural status (ESCS). The level-2 predictor is student-to-teacher-ratio (STR). As missing values were found in the growth mindset, ESCS, and STR variables, the comprised samples became 9,196 students and 308 schools.

Socio-demographic variables were gender (female or male) and ESCS. The students' ESCS is derived from several variables relating to home and background information of students and ranged from -8.17 to 4.21 (OECD, 2016).

Growth mindset was assessed by responses toward the following statement: "Your intelligence is something about you that you can't change very much." Responses were coded as 1: Strongly disagree, 2: Disagree, 3: Agree, 4: Strongly agree. As the growth mindset represents belief that an individual can improve his/her abilities over time, then the coded responses must be reversed from 1: Strongly agree up to 4: Strongly disagree. As a result, 1 indicates the lowest growth mindset score, while 4 indicates the highest growth mindset score for each individual. In this study, the growth mindset is treated as a continuous predictor in order to have insightful interpretation.

Data Analysis

A total of 9,196 students and 308 schools with all variables of interests were used to investigate the effects of gender, growth mindset, students' ESCS, and STR on mathematics achievement by fitting two-level multilevel models. Descriptive statistics were calculated to provide information about the sample characteristics. The parameters of the multilevel model were estimated using the restricted maximum likelihood (REML) method through the use of lme() function in R statistical software version 3.6.0 from nlme package (Pinheiro et al., 2021). Tree random intercept models and one random slope model were fitted to the data. The Akaike information criterion (AIC) and Bayesian information criterion (BIC) were used to select the best model, where lower value indicated a more parsimonious model. Since the PISA 2018 data consists of school-level data and students-level data, in this study, we used the multilevel analysis as follows.

Random Intercept Model

A random-intercept model is a simple multilevel model with only one random level-1 coefficient (Finch et al., 2019). The level-1 of the random intercept model is given as,

$$y_{ij} = \beta_{0j} + \varepsilon_{ij} \qquad (1)$$

The level-2 of the random intercept model is expressed as

$$\beta_{0j} = \gamma_{00} + U_{0j} \qquad (2)$$

where the *ij* subscript refers to the *i*th student in the *j*th school, y_{ij} represents the mathematics achievement score for the ith student in the jth school, ε_{ij} is assumed to follow a normal distribution with a mean of zero and a constant variance of σ^2, $\varepsilon_{ij} \sim N(0, \sigma^2)$. Model (1) predicts the mathematics achievement from just an intercept which allows to vary randomly within school. The γ_{00} represents an average or general intercept value that holds across schools, U_{0j} is a school-specific effect on the intercept assuming a normal distribution with a mean of zero and a constant variance value denoted as τ_{00}, $U_{0j} \sim N(0, \tau_{00})$. The γ_{00} is a fixed effect because it remains constant across all schools, and U_{0j} is a random effect because it varies between schools. It is assumed that both variances τ_{00} and σ^2 are uncorrelated. Model (2) allows the intercept differs across schools which leads to the random intercept. Model (1) and (2) can be combined as

$$y_{ij} = \gamma_{00} + U_{0j} + \varepsilon_{ij} \qquad (3)$$

Model (3) is also known as an unconditional mean or null or empty model in the multilevel modelling context.

In Equation (1), the mathematics achievement score of students i in school j (y_{ij}) is modelled as a function of the mean score in mathematics achievement for school j (β_{0j}) plus a residual term that reflects individual student differences around the mean of school j (ε_{ij}). In Equation (1), the mathematics achievement mean score for school j (β_{0j}) is modelled as a function of a grand-mean score in mathematics achievement (γ_{00}) plus a school-specific deviation from the grand mean (U_{0j}). Equation (3) is important for researchers since it facilitates the writing of the syntax commands on statistical software packages.

As the students are clustered within a school unit, the correlation among students' scores within school structure can be derived using the intraclass correlation (ICC) which is expressed as,

$$ICC = \frac{\tau_{00}}{\tau_{00}+\sigma^2} \quad (4)$$

The ICC is a measure of proportion of variation in the outcome variable that occurs between groups versus the total variation present. ICC values between 0.05 and 0.20 are common in multilevel modelling in social research studies (Peugh, 2010). The need for a multilevel analysis is not only based on a non-zero ICC but also the design effect. The design effect is used to justify for accounting the multilevel structure in the analysis (Maas & Hox, 2005). The design effect is determined by

$$Design\ effect = 1 + \frac{(n_c - 1)}{ICC} \quad (5)$$

where n_c is the number of students per school. The value of design effect estimates greater than 2.0 indicate a need for multilevel modelling (Muthén, 1991; 1994; Muthén & Satorra, 1995).

Adding three predictors of level-1 in the random intercept model is extending the empty model into four equations in the level-2. The scales of two continuous variables are centered around the grand mean. Thus, the grand mean centering variables with zero values represent the overall mathematics achievement mean score across all schools. The level-1 of the random intercept model with three predictors of level-1 is expressed as,

$$y_{ij} = \beta_{0j} + \beta_{1j}x_{1ij} + \beta_{2j}(x_{2ij} - \bar{x}_{2..}) + \beta_{3j}(x_{3ij} - \bar{x}_{3..}) + \varepsilon_{ij} \quad (6)$$

The level-2 of the random intercept model with three predictors of level-1 is formulated as,

$$\beta_{0j} = \gamma_{00} + U_{0j}$$
$$\beta_{1j} = \gamma_{10}$$
$$\beta_{2j} = \gamma_{20}$$
$$\beta_{3j} = \gamma_{30} \quad (7)$$

Combining Eq. (6) and (7) yield,

$$y_{ij} = \gamma_{00} + \gamma_{10}x_{1ij} + \gamma_{20}(x_{2ij} - \bar{x}_{2..}) + \gamma_{30}(x_{3ij} - \bar{x}_{3..}) + U_{0j} + \varepsilon_{ij} \quad (8)$$

The first term in Equation (8) is identical to Equation (2) where γ_{00} is the grand mean and U_{0j} is a residual that allows the mathematics achievement mean scores to vary across schools. Equation (6) illustrates the definition of a fixed effect model in level-1: the impact of gender (x_{1ij}, 1 for male and 0 for female), growth mindset ($(x_{2ij} - \bar{x}_{2..})$, the growth mindset of student i at the school j is centered around the grand mean), and students' ESCS ($(x_{3ij} - \bar{x}_{3..})$, the ESCS of student i at the school j is centered around the grand mean) on mathematics achievement across each school (β_{1j}, β_{2j} and β_{3j}

respectively) are captured by single estimates that express the average effect of gender, growth mindset, and students' ESCS on mathematics achievement across all schools (γ_{10}, γ_{20} and γ_{30} respectively).

The addition of three predictors of level-1 and one predictor of level-2 in the random intercept model is expressed as,

$$y_{ij} = \beta_{0j} + \beta_{1j}x_{1ij} + \beta_{2j}(x_{2ij} - \bar{x}_{2..}) + \beta_{3j}(x_{3ij} - \bar{x}_{3..}) + \beta_{4j}(w_j - \bar{w}_.) + \varepsilon_{ij} \quad (9)$$

where w_j is the predictor at level-2 which represents the STR at school j. The predictor w_j is also centered around its grand mean.

The level-2 of the random intercept model with three predictors of level-1 and one of predictor level-2 is formulated as,

$$\beta_{0j} = \gamma_{00} + U_{0j}$$
$$\beta_{1j} = \gamma_{10}$$
$$\beta_{2j} = \gamma_{20}$$
$$\beta_{3j} = \gamma_{30}$$
$$\beta_{4j} = \gamma_{40} \quad (10)$$

Combining Equations (9) and (10) yield,

$$y_{ij} = \gamma_{00} + \gamma_{10}x_{1ij} + \gamma_{20}(x_{2ij} - \bar{x}_{2..}) + \gamma_{30}(x_{3ij} - \bar{x}_{3..}) + \gamma_{40}(w_j - \bar{w}_.) + U_{0j} + \varepsilon_{ij} \quad (11)$$

where the addition of β_{4j} represents the impact of STR on mathematics achievement across each school that is captured by a single estimate of γ_{40}.

Random Slope Model

A random slope model involves the mean scores from each of many schools as an outcome to be predicted by group characteristics (Raudenbush & Bryck, 2002). The random slope model in this study uses Equation (9) as the level-1model, whereas the level-2 model is given as follows,

$$\beta_{0j} = \gamma_{00} + U_{0j}$$
$$\beta_{1j} = \gamma_{10}$$
$$\beta_{2j} = \gamma_{20}$$
$$\beta_{3j} = \gamma_{30}$$
$$\beta_{4j} = \gamma_{40} + U_{1j} \quad (12)$$

where there is one level-2 predictor w_j (STR). Substituting Equation (12) into Equation (9) yield the combined model as follows,

$$y_{ij} = \gamma_{00} + \gamma_{10}x_{1ij} + \gamma_{20}(x_{2ij} - \bar{x}_{2..}) + \gamma_{30}(x_{3ij} - \bar{x}_{3..}) + \gamma_{40}(w_j - \bar{w}_.) + U_{0j} + U_{1j}(w_j - \bar{w}_.) + \varepsilon_{ij} \quad (13)$$

where $\begin{bmatrix} U_{0j} \\ U_{1j} \end{bmatrix} \sim N(\mathbf{0}, \boldsymbol{\tau})$, $\boldsymbol{\tau} = \begin{bmatrix} \tau_{00} & \tau_{10} \\ \tau_{01} & \tau_{11} \end{bmatrix}$, and $\varepsilon_{ij} \sim N(0, \sigma^2)$. The τ_{00} is the variance in intercepts between schools (and the level 2 variance at STR equals to 0), τ_{11} is the variance in slopes (STR) between schools, and τ_{01} or τ_{10} is the covariance between intercepts and slopes. Finally, four combined models for multilevel analysis are developed in this study, namely the models in Equation (3), (8), (11), and (13).

RESULTS AND DISCUSSION

Descriptive Statistics

Students' mean score in mathematics was 399.98 with a standard deviation of 79.91, while approximately 53 percent of students' mathematics scores were below the mean. Given that PISA standardized the mathematics score with an average of 487 (SD=89) across the OECD countries, Indonesian students appeared to perform worse than the OECD average. Table 1 shows that more boys than girls in Indonesia had mathematics scores below the mean. The students with higher belief that growth mindset can improve their abilities over time had much larger mathematics mean scores than those with less belief.

The range of students ESCS variable was between -5.78 and 2.97 where its mean score was -1.40 and standard deviation was 1.11, while the student-to-teacher ratio variable ranged from 1.54 to 100 where its mean score was 18.19 and standard deviation was 7.46. Although the Pearson´s correlation between the students' ESCS and the mathematics score was low, it showed that the two variables had a significant relationship. Also, there was a significant relationship between the student-to-teacher ratio and the mathematics score.

Table 1. Gender, Growth Mindset, Student-to-Teacher Ratio, and Mathematics Achievement

Variable	N (%)	Mean mathematics score (SD)	Percentage of students with maths below the means
Gender			
Boys	4,495 (48.88%)	394.85 (80.65)	53.26%
Girls	4,701 (51.12%)	404.87 (78.90)	52.80%
Growth Mindset			
1	1,991 (21.65%)	380.68 (69.06)	52.59%
2	4,077 (44.33%)	382.38 (71.76)	53.50%
3	2,199 (23.91%)	436.30 (81.38)	50.52%
4	929 (10.10%)	432.53 (91.64)	48.65%
Student ESCS	-	0.39 (<0.0001)*	-
Student-to-Teacher Ratio	-	0.05 (<0.0001)*	-

Note: *Pearson´s correlation coefficient (p-value), N(%) represents the total of students in each category of the variable, SD represents the standard deviation.

Two-Level Multilevel Models

In this study, the multilevel analysis was developed in a few steps, starting with the most straightforward model and gradually moving to a more complex model. The scales variables were centered on the grand mean for the purposes of this analysis.

Step 1: Model without Fixed Predictors (MWFP)

Equation (3) represents the simplest model that considers school effects on the mathematics achievement. The multilevel model shown in Equation (3) was estimated and results are shown in the first column of Table 2. A significant non-zero grand-mean score in mathematics achievement was observed, $\hat{\gamma}_{00} = 390.63, p < 0.001$. The level-1 variance component estimate shows the magnitude of mathematics achievement score variation across students within a school, $\hat{\sigma}^2 = 2{,}641.25$. The variance component in the mathematics achievement mean scores across schools was $\hat{\tau}_{00} = 3{,}717.04$. The estimated ICC can be obtained by substituting these two-variance component estimates in the following equation:

$$\widehat{ICC} = \hat{\tau}_{00}/(\hat{\tau}_{00} + \hat{\sigma}^2) = 3{,}717.04/(3{,}717.04 + 2{,}641.25) = 0.58.$$

The ICC estimate showed that 58% of the mathematics achievement variance occurred across schools. The average number of students per school in the PISA 2018 Indonesian dataset was n_c = 9,196/308 = 29.86. The design effect estimate is computed by:

$$\widehat{Design\,Effect} = 1 + (n_c - 1)\widehat{ICC} = 1 + (29.86 - 1)(0.58) = 17.74.$$

The ICC estimate of 58% (>0%) and the design effect estimate of 17.74 (>2) indicate the clear need for multilevel modelling of mathematics achievement data.

Step 2: Adding Student-Level Predictor to Model (Level 1: Fixed)

The multilevel model shown in Equation (8) was estimated and results are shown in the second column of Table 2. Results again showed a significantly non-zero mean score in mathematics achievement ($\hat{\gamma}_{00} = 392.23, p < 0.001$). All of the level-1 predictors included in the model were found to be statistically significant predictors of students' achievement in mathematics where it was moderately significant for the gender effect ($\hat{\gamma}_{10} = -1.89, p < 0.10$). More specifically, boys had underperformed in mathematics compared with girls. The first regression slope indicates that mathematics achievement increases in growth mindset associated with approximately an eight-point increase in achievement, on average ($\hat{\gamma}_{20} = 8.43, p < 0.001$). Meanwhile, the second regression slope shows that about a three-point increase in mathematics achievement on average was associated with an increase in students' ESCS ($\hat{\gamma}_{30} = 3.07, p < 0.001$). By including all these level-1 predictors in the multilevel model, the between-school variance component (τ_{00}) considerably decreased from 3,717.04 to 3,308.57. This suggests that much of the variance between schools was attributable to the students' background and growth mindset. The variance within schools also declined from 2,641.25 to 2,589.72.

As the level-two variance component was about 11% explained by the student-level variables in this model, it was also important to find the predictor at the school level.

Table 2. Model Summaries of Several Multilevel Models. MWFP denotes the model without fixed predictors (Equation 3); level-1: fixed denotes the random intercept model with the fixed level-1 predictors (Equation 8), level-1 & 2: fixed denotes the random intercept model with fixed level-1 and 2 predictors (Equation 11), and level-2 denotes the random slope model with fixed level-1 and 2 predictors and random level-2 predictor (Equation 13)

Parameters	***MWFP***	**Level-1: fixed**	**Level-1 & 2: fixed**	**Level-2: random**
Regression coefficients (fixed effects)				
Intercept (γ_{00})	390.63 (3.54)***	392.23 (3.39)***	393.42 (3.38)***	393.77 (3.42)***
Gender (male)	-	-1.89 (1.12)•	-1.85 (1.12)•	-1.85 (1.12)•
Growth Mindset	-	8.43 (0.62)***	8.43 (0.62)***	8.43 (0.62)***
Student ESCS	-	3.07 (0.63)***	3.07 (0.63)***	3.07 (0.63)***
Student-to-Teacher Ratio	-	-	1.08 (0.38)**	1.29 (0.42)**
Variance components (random effects)				
Residual (σ^2)	2,641.25 (39.62)	2,589.72 (38.85)	2,589.57 (38.85)	2,589.62 (38.85)
Intercept (τ_{00})	3,717.04 (310.70)	3,308.57 (277.48)	3,236.68 (272.10)	3,257.23 (302.21)
Slope (τ_{11})	-	-	-	0.21
Correlation (τ_{01})	-	-	-	0.56
Information criteria				
AIC	99,669.55	99,458.94	99,453.04	99,456.36
BIC	99,690.93	99,501.69	99,502.92	99,520.49

Parameter estimate and standard errors listed in parentheses.
*** $p < 0.001$, ** $p < 0.01$, * $p < 0.05$, •$p < 0.10$

Step 3: Adding School-Level Predictor to Model (Level 1 and 2: Fixed)

Having analyzed the variables at the student level and finding that there was still a lot of unexplained variation at the school level, the next step was to determine whether the student-to-teacher ratio could explain the remaining variation between schools. The results of implementing the multilevel analysis given in Equation (11) shows that the student-to-teacher ratio was found to be a statistically significant predictor of the mathematics achievement with the estimated coefficient ($\hat{\gamma}_{40} = 1.08, p < 0.01$) which is about three times its standard error (SE=0.38). Results again showed a significantly non-zero mean score in mathematics achievement ($\hat{\gamma}_{00} = 393.42, p < 0.001$) whereas all of the level-1 predictors were found to be statistically significant predictors of students' mathematics achievement except for gender. The positive value of the coefficient suggests that there was a large score in mathematics achievement among the students who study at schools with high numbers of STR. The inclusion of the STR variable in the model resulted in a slight reduction in the between-school variance from 3,308.57 to 3,236.68, indicating that about 2 percent of the between-school difference in mathematics achievement was explained by the school's mean STR attended by students, whereas a small decrease 2,589.72 to 2,589.57 was seen in the variation within schools.

In Equation (11), the level-2 predictor was assumed as a fixed effect. Moreover, Equation (13) allows the impact of STR on mathematics achievement to vary from one school to another. The multilevel model shown in Equation (13) was estimated and results are shown in the fourth column of Table 2. The STR is statistically significant related to the mathematics achievement ($\hat{\gamma}_{40} = 1.29, p < 0.01$). The variation across schools is due to the impact of STR on the mathematics achievement ($\hat{\tau}_{11}$) is 0.21, suggesting that the coefficient does not differ too much between schools. The unexplained variation at level 2 increased slightly from 3,236.68 to 3,257.23, as well as the unexplained variation at level 1 from 2,589.57 to 2,589.62. The interaction between gender and growth mindset was tested but not shown in the method section and found to have no significant effect on mathematics achievement ($p = 0.13$). This means that the growth mindset is the same for both boys and girls.

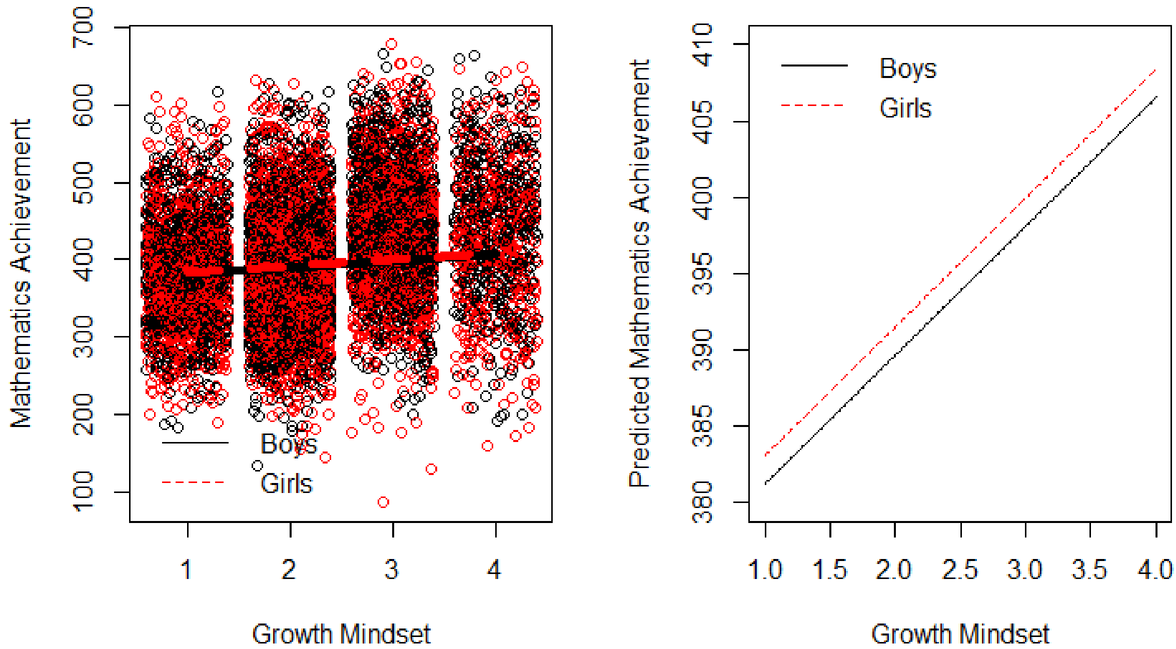
Comparing across models, the multilevel model in Equation (11) is selected as the best model as it has the lowest AIC (AIC = 99,453.04) in comparison with the other three models. Although, the BIC of the multilevel model in Equation (11) is similar to model in Equation (8), the significant coefficient of STR suggests that this variable should be included in the model. The final estimated model is given as follows,

$$\widehat{Math}_{ij} = 393.42 - 1.85Male_{ij} + 8.43Growth_{ij} + 3.07ESCS_{ij} + 1.08STR_j + \hat{U}_{0j}$$

with the two variance components of $\hat{\tau}_{00} = 3{,}236.68, \hat{\sigma}^2 = 2{,}589.57$. It is important to remember that the last three independent variables are mean centered.

Figure 1(a) shows a significant increase in mathematics achievement as students' growth mindset increased. Fitting two levels multilevel model in Equation (11) without grand mean centering yields Figure 1(b). Figure 1(b) suggests that the mathematics achievement scores increase as the students' growth mindset increases, where girls perform better than boys.

The MWFP (Equation 3) suggested that about 58% of the variance was attributed to differences between school and 42% to differences within school. The final multilevel (Equation 11) model reveals that girls in Indonesia outperformed boys in mathematics achievement. This finding is supported by previous research which found that girls had scored 10 points higher than boys in mathematics achievement based on the 2015 Trends in International Mathematics and Science Study (TIMSS) (Luschei, 2017). The significance of gender to the students' mathematics achievement was also revealed in previous PISA 2012 result (Pakpahan, 2016).



(a) (b)

Figure 1. (a) The observed student growth mindset versus mathematics achievement at the averages of students' ESCS and STR for each gender along with the two regression lines, (b) The student growth mindset versus predicted mathematics achievement at the averages of students' ESCS and STR for each gender

The findings of this study suggested that growth mindset, students' ESCS and STR are important factors to predict the mathematics achievement. This is in line with what Dweck and Yeager (2019) proposed, namely that believing in our effort is a positive thing that can help us grow instead of a negative thing that leads to deficient ability. More specifically, the finding of positive relationship between the growth mindset and mathematics achievement is in line with the result from Boaler et al. (2018). They found that students with higher growth mindset showed that their mathematical perceptions improved as the subject became more interesting and seen a creative subject.

The finding on how the increased students' ESCS yields higher mathematics achievement is in accordance with the previous studies from Incikabi et al. (2012), Thien and Ong (2015) as well as Cheng and Hsu (2016). This result suggests that when discussing about mathematics achievement, social economic status should be addressed. This situation can lead to the inequality in mathematics achievement between the students' higher and lower socioeconomic status.

The finding on how the increased STR yields higher mathematics achievement is contrary to the finding of Koc and Celik (2015) which pointed out that cities in Turkey with greater number of students per teacher tend to have a lower achievement. This result can be explained by the fact that highly successful schools in Indonesia tend to have higher student-to-teacher ratio as they have more students with relatively similar numbers of teachers compared to other schools. For example, a highly successful school can have 1,560 students with 58 teachers, compared to a non-highly successful school with 400 students and 30 teachers.

The predictors of student-level and school-level in the final model contributed to a decrease in the unexplained variation between schools. More specifically, the final equation model explained the 13% of the unexplained variance at the school-level. The student-level variables especially student growth mindset, made a great contribution to the decrease of the level-two unexplained variance components besides the school-level variable of STR.

In contrast with the PISA 2015 which focused on mathematics ability, the main focus of PISA 2018 was reading ability. As a consequence, the recent PISA test does not accommodate many specific variables related to mathematics. To obtain more information, we could sum up the results of two consecutive PISA tests as follows. In the student level, factors that contribute to their achievement in mathematics are growth mindset, gender, students' social-economic status, mathematics self-efficacy, attitudes toward learning outcomes, anxiety, mathematics self-concept, mathematics behavior (Thien et al., 2015), and students' sense of belonging toward mathematics (Kartianom & Ndayizeye, 2017). In the school level, the student-to-teacher ratio, average of socio-economic status (Kartianom & Ndayizeye, 2017), average students' sense-of-belonging, average openness to problem-solving, and average mathematics efficacy (Thien et al., 2015) affects the student performance in mathematics. To confirm the significance of these factors to student performance, an assessment at the national level should be conducted. Otherwise, we may wait until the next PISA is held with a focus on mathematical ability.

It is also important to note that the PISA test in Indonesia was done on various school levels. A large sample has been taken from two provinces, namely DKI Jakarta and DI Yogyakarta so that the result from these areas can be compared to the result from other areas in Indonesia as well as other countries (Avvisati et al., 2019). For future research, more variables can be analyzed, with special attention to the differences between Indonesian provinces.

This study implies that growth mindset can be considered as important part of policy making especially in Indonesia. As Indonesian students achieve lower in Mathematics than other students in other countries, growth mindset can be used for increasing the score, as well as to be more competitive students. Two policies can be considered for government and universities, such as providing a large-scale intervention for increasing and maintaining growth mindset among students, and providing a curriculum design which includes growth mindset as a part of teaching mathematics. The former can be done by recommending these results to the government institution called the Center for the Development and Empowerment of Educators and Education Personnel for Mathematics to include this insight for teaching mathematics through their regular training program for mathematics teachers. The program can be quite short, done online and scalable as conducted by Yeager et al. (2019) in the United States. The latter can be recommended on the online sharing platform managed by government to share how important growth mindset in predicting mathematics achievement, especially among Indonesian students. By sharing this insight, teachers can understand the role of growth mindset in teaching mathematics and apply that into the classroom settings. These two recommendations can at least give a shine on how to increase mathematics achievement. Besides those, government should consider how important socio-economic status and school condition for predicting students' mathematics achievement. However, other factors that affect the low performance in mathematics should be addressed at the national level in order to increase the mathematics ability of the students. Even though the results of our study are not conclusive, some of the factors that influence the low performance on mathematics in the PISA 2018 test can be considered for the policy makers in education of the country to start fixing this complex problem, if we want to produce literate people that can help in fully participating in the society.

CONCLUSIONS

Programme for International Students Assessment (PISA) is a large-scale assessment in mathematics, science, and reading ability for students in various countries. The PISA 2018 showed that Indonesian students exhibited low performance in mathematics. A multilevel analysis shows that several student-level factors contributed significantly to this result, namely gender, growth mindset, as well as students' economic and socio-cultural status (ESCS). Girls outperform boys in mathematics achievement. As the student growth mindset increases, so does the students' mathematics achievement, as well as the students' ESCS. At the school level, we found that a higher student-to-teacher ratio is related to higher students' mathematical performance. In sum, students' mathematics achievement should be seen within the context of psychological, social and school factors, instead of merely about teaching mathematics. Further research is needed to grasp the effect of other variables, as well as comparing the results between schools in different provinces in Indonesia. Even then, these results can help policy makers in education of Indonesia to address this problem for the present and future development of the society.

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